Numerical analysis of a coupled pair of Cahn-Hilliard equations

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A Thesis presented for the degree of
Doctor of Philosophy

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Dedicated to my teachers:

Alinius AN, Abdul Muis Samah, Masrizal,
Ijing Padli Supeno, Azis Jarah, Muchtar Rahman,
Kurt Georg, Eigen L. Algower
from whom I have learned many very beautiful things.

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Abstract

A mathematical analysis has been carried out for a coupled pair of Cahn-Hilliard equations, which appear in modelling a phase separation on a thin film of binary liquid mixture coating substrate, which is wet by one component. Existence and uniqueness are proved for a weak formulation of the problem, which possesses a Lyapunov functional. Regularity results are presented for the weak formulation.

A fully practical piecewise linear finite element approximation is proposed where existence and uniqueness of the numerical solution, and its convergence to the solution of the continuous problem are proven. An error bound between the discrete and continuous solutions is given in three space dimensions. A practical algorithm for solving the resulting algebraic problem at each time step is suggested and its convergence is proven. Finally, linear stability analysis for one space dimension is presented, and some numerical simulations in one and two spaces dimension are exhibited.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, the Department of Mathematical Sciences, the University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

Let Ω be bounded domain in $\mathbb{R}^d (d \leq 3)$ with Lipschitz boundary $\partial\Omega$. We consider a coupled pair of Cahn-Hilliard Equations modelling a phase separation on a thin film of binary liquid mixture coating substrate, which is wet by one component denoted by A and the other by B (see [25]):

Find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} = \Delta w_1 \quad \text{in } \Omega, t > 0, \tag{1.0.1a}$$

$$\frac{\partial u_2}{\partial t} = \Delta w_2 \quad \text{in } \Omega, t > 0, \tag{1.0.1b}$$

where

$$w_1 = \frac{\delta F(u_1, u_2)}{\delta u_1}, \tag{1.0.1c}$$

$$w_2 = \frac{\delta F(u_1, u_2)}{\delta u_2}, \tag{1.0.1d}$$

$$\begin{aligned} F(u_1, u_2) &= b_1 u_1^4 - a_1 u_1^2 + c_1 |\nabla u_1|^2 \\ &\quad + b_2 u_2^4 - a_2 u_2^2 + c_2 |\nabla u_2|^2 \\ &\quad + D \left(u_1 + \sqrt{\frac{a_1}{2b_1}} \right)^2 \left(u_2 + \sqrt{\frac{a_2}{2b_2}} \right)^2 . \end{aligned} \tag{1.0.1e}$$

Here $\delta F(u_1, u_2)/\delta u_i$, for $i = 1, 2$, indicates the functional derivative. The variable u_1 denotes a local concentration of A or B and u_2 indicates the presence of a liquid or a vapour phase. The constant c_i denotes the surface tension of u_i. The coefficient a_i is proportional to $T_{c_i} - T$, where T_{c_1} corresponds to the critical temperature of the A-B phase separation, and T_{c_2} represents the critical temperature of the liquid-vapour phase separation.

If $a_1 > 0$, $a_2 > 0$, there are two equilibrium phases for each field corresponding to $u_1 = \pm\sqrt{\frac{a_1}{2b_1}}$ and $u_2 = \pm\sqrt{\frac{a_2}{2b_2}}$, denoted u_1^+, u_1^-, u_2^+, and u_2^-, respectively. The coupling D energetically inhibits the existence of the phase denoted by the (u_1^+, u_2^+). Thus we have a three-phase system: liquid A corresponds to (u_1^-, u_2^-) regions, liquid B to (u_1^+, u_2^-) regions and the vapour phase to (u_1^-, u_2^+) regions.

To simplify the presentation, as in [25], we choose the values in (1.0.1e) as follows:

$$b_1 = b_2 = \frac{1}{4}, \quad a_1 = a_2 = \frac{1}{2}, \quad c_1 = c_2 = \frac{\gamma}{2},$$

namely

$$F(u_1, u_2) = \psi(u_1) + \frac{\gamma}{2}|\nabla u_1|^2 + \psi(u_2) + \frac{\gamma}{2}|\nabla u_2|^2 + 2D\Psi(u_1, u_2), \tag{1.0.2a}$$

where

$$\psi(r) = \frac{1}{4}(r^2 - 1)^2, \tag{1.0.2b}$$
$$\Psi(r, s) = \frac{1}{2}(r + 1)^2(s + 1)^2, \tag{1.0.2c}$$

although all of the results may be modified to the general case. Here $D > 0$ and $\gamma > 0$ are prescribed constants. Together with this problem we include the following boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.0.3a}$$
$$u_1(x, 0) = u_1^0(x), \quad u_2(x, 0) = u_2^0(x) \quad \text{on } \Omega, \tag{1.0.3b}$$

where ν is the unit normal pointing out of Ω.

Thus the problem now is to find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} = \Delta w_1 \quad \text{in } \Omega,\, t > 0, \tag{1.0.4a}$$

$$\frac{\partial u_2}{\partial t} = \Delta w_2 \quad \text{in } \Omega,\, t > 0, \tag{1.0.4b}$$

where

$$w_1 = \frac{\delta F(u_1, u_2)}{\delta u_1} = \phi(u_1) - \gamma \Delta u_1 + 2D\Psi_1(u_1, u_2), \tag{1.0.4c}$$

$$w_2 = \frac{\delta F(u_1, u_2)}{\delta u_2} = \phi(u_2) - \gamma \Delta u_2 + 2D\Psi_2(u_1, u_2), \tag{1.0.4d}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.0.4e}$$

$$u_1(x,0) = u_1^0(x), \qquad u_2(x,0) = u_2^0(x), \tag{1.0.4f}$$

$$\phi(r) = \psi'(r), \tag{1.0.4g}$$

$$\Psi_1(r,s) = \frac{\partial \Psi(r,s)}{\partial r}, \tag{1.0.4h}$$

$$\Psi_2(r,s) = \frac{\partial \Psi(r,s)}{\partial s}. \tag{1.0.4i}$$

If $D = 0$, the problem reduces to two decoupled Cahn-Hilliard equations, which has been discussed at length in the mathematical literature; for reviews see [18, 20, 31]. For this type of problem, we do not have liquid-vapour interfaces.

To obtain a weak formulation of the problem above, let V be the trial space, that is,

$$V = \left\{ v : \int_\Omega (|\nabla v|^2 + v^2) dx < \infty \right\}.$$

Multiply (1.0.4a) and (1.0.4c) by any test function $v \in V$, integrate over Ω and rearrange the terms to get

$$\left(\frac{\partial u_1}{\partial t}, v \right) = (\Delta w_1, v), \tag{1.0.5a}$$

$$(w_1, v) = (\phi(u_1), v) - \gamma(\Delta u_1, v) + 2D(\Psi_1(u_1, u_2), v); \tag{1.0.5b}$$

similarly from (1.0.4b) and (1.0.4d) we obtain

$$\left(\frac{\partial u_2}{\partial t}, v\right) = (\Delta w_2, v), \tag{1.0.5c}$$

$$(w_2, v) = (\phi(u_2), v) - \gamma(\Delta u_2, v) + 2D(\Psi_2(u_1, u_2), v). \tag{1.0.5d}$$

Applying Green's formula

$$\int_\Omega \Delta u v dx = \int_\Gamma v \frac{\partial u}{\partial \nu} dx - \int_\Omega \nabla u \nabla v dx \quad \forall v \in C^1(\bar{\Omega}), \tag{1.0.6}$$

to the terms containing the Laplacian in (1.0.5a–d) and using boundary conditions (1.0.4e), we obtain the weak formulation

(P) Find $\{u_1, u_2, w_1, w_2\} \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega),\ t \in [0, T]$ such that $\forall \eta \in H^1(\Omega)$

$$\left(\frac{\partial u_1}{\partial t}, \eta\right) = -(\nabla w_1, \nabla \eta), \tag{1.0.7a}$$

$$(w_1, \eta) = (\phi(u_1), \eta) + \gamma(\nabla u_1, \nabla \eta) + 2D(\Psi_1(u_1, u_2), \eta), \tag{1.0.7b}$$

$$u_1(x, 0) = u_1^0(x), \tag{1.0.7c}$$

and

$$\left(\frac{\partial u_2}{\partial t}, \eta\right) = -(\nabla w_2, \nabla \eta), \tag{1.0.7d}$$

$$(w_2, \eta) = (\phi(u_2), \eta) + \gamma(\nabla u_2, \nabla \eta) + 2D(\Psi_2(u_1, u_2), \eta), \tag{1.0.7e}$$

$$u_2(x, 0) = u_2^0(x). \tag{1.0.7f}$$

We now give a brief description of the content of this thesis. In Chapter 2 a global existence and uniqueness theorem for a weak formulation possessing a Lyapunov function is proven. Regularity results are presented for the weak formulation.

In Chapter 3 we prove interpolation error estimates in the finite element space as tools for analysis in Chapter 3 and 4. Then a semidiscrete finite element approximation is proposed where the existence and uniqueness are proven. Also an error bound between the semidiscrete and continuous solutions is given.

In Chapter 4 two fully discrete finite element approximation are proposed where the existence and uniqueness are proven. The convergence of the discrete to the continuous solutions is shown for Scheme 1. An error bound between the discrete and continuous solutions is also proven for Scheme 1.

In Chapter 5 two practical algorithms (implicit and explicit methods) for solving the finite element approximation at each time step are suggested. We discuss the convergence theory for the implicit scheme, which is used to solve the system arising from Scheme 1. We also discuss in this chapter some computational results for one and two space dimensions. We use the implicit scheme for all simulations. Before showing some computational results, we discuss linear stability solutions in one space dimension.

Chapter 2

Evolutionary Problem

In this chapter a global existence and uniqueness theorem for a weak formulation possessing a Lyapunov function is proven. Regularity results are presented for the weak solution.

2.1 Notation

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \leq 3$ with boundary $\partial\Omega$. For $d = 2, 3$ we assume that $\partial\Omega$ is a Lipschitz boundary. Throughout this thesis we adopt the standard notation for Sobolev spaces, denoting the norm of $W^{m,p}(\Omega)$ ($m \in \mathbb{N}, p \in [1, \infty]$) by $\|\cdot\|_{m,p}$ and semi-norm by $|\cdot|_{m,p}$. For $p = 2$, $W^{m,p}(\Omega)$ will be denoted by $H^m(\Omega)$ with the associated norm and semi-norm written as $\|\cdot\|_m$ and $|\cdot|_m$, respectively. In addition we denote the $L^2(\Omega)$ inner product over Ω by $(\cdot, \cdot)$ and define the mean integral by

$$\fint \eta := \frac{1}{|\Omega|}(\eta, 1) \quad \forall \eta \in L^1(\Omega).$$

We also use the following notation, for $1 \leq q < \infty$,

$$L^q(0,T; W^{m,p}(\Omega)) := \left\{ \eta(x,t) : \eta(\cdot,t) \in W^{m,p}(\Omega), \int_0^T \|\eta(\cdot,t)\|_{m,p}^q \, dt < \infty \right\},$$

$$L^\infty(0,T; W^{m,p}(\Omega)) := \left\{ \eta(x,t) : \eta(\cdot,t) \in W^{m,p}(\Omega), \operatorname*{ess\,sup}_{t \in (0,T)} \|\eta(\cdot,t)\|_{m,p} < \infty \right\},$$

$$\|\chi\|_{L^q(0,T;W^{m,p}(\Omega))} = \begin{cases} \left(\int_0^T \|\chi(\cdot,t)\|_{m,p}^q dt\right)^{1/q} & \text{for } 1 \le q < \infty, \\ \operatorname*{ess\,sup}_{t\in(0,T)} \|\chi(\cdot,t)\|_{m,p} & \text{for } q = \infty. \end{cases}$$

We introduce the Green's operator $\mathcal{G} : \mathcal{F} \mapsto V$ approximating the inverse Laplacian with zero Neumann boundary data defined by

$$(\nabla \mathcal{G} v, \nabla \eta) = \langle v, \eta \rangle \quad \forall \eta \in H^1(\Omega), \tag{2.1.1}$$

where

$$\mathcal{F} := \{\eta \in (H^1(\Omega))' : \langle \eta, 1 \rangle = 0\},$$
$$V := \{\eta \in H^1(\Omega) : (\eta, 1) = 0\},$$

and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ such that

$$\langle v, \eta \rangle := (v, \eta) \quad \forall \eta \in L^2(\Omega).$$

The existence and uniqueness of $\mathcal{G}v$ follows from the Lax-Milgram theorem (see Appendix) and the Poincaré inequality

$$|\xi|_{0,p} \le \widetilde{C}_P(|\xi|_{1,p} + |(\xi, 1)|) \quad \forall \xi \in W^{1,p}(\Omega), \quad p \in [1, \infty]. \tag{2.1.2}$$

We define a norm on $\mathcal{F}$ as

$$\|v\|_{-1} := |\mathcal{G}v|_1.$$

Note that for $v \in \mathcal{F} \cap L^2(\Omega)$, we have

$$\|v\|_{-1}^2 = (\nabla \mathcal{G}v, \nabla \mathcal{G}v) = \langle v, \mathcal{G}v \rangle = (v, \mathcal{G}v) = (\mathcal{G}v, v), \tag{2.1.3}$$

and noting the Young inequality, for $\epsilon > 0$, $a, b \geq 0$ and $1 < p < \infty$

$$ab \leq \epsilon \frac{a^p}{p} + \epsilon^{-q/p} \frac{b^q}{q}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \tag{2.1.4}$$

we obtain for all $\alpha > 0$ with $\epsilon = \alpha$, and $p = q = 2$,

$$(v, v) = (\nabla \mathcal{G} v, \nabla v) = (\nabla v, \nabla \mathcal{G} v) \leq |v|_1 \|v\|_{-1} \leq \frac{\alpha}{2} |v|_1^2 + \frac{1}{2\alpha} |\mathcal{G} v|_1^2. \tag{2.1.5}$$

Using the Poincaré inequality (2.1.2), the Cauchy-Schwarz inequality and (2.1.3) we obtain

$$\|v\|_{-1}^2 = (\nabla \mathcal{G} v, \nabla \mathcal{G} v) = \langle \mathcal{G} v, v \rangle \leq |\mathcal{G} v|_0 |v|_0 \leq \tilde{C}_P |\mathcal{G} v|_1 |v|_0. \tag{2.1.6}$$

For later purposes, we recall the Hölder inequality for $u \in L^p$, $v \in L^q$ and $1 < p < \infty$,

$$\int_\Omega |uv| dx \leq \left(\int_\Omega u^p dx \right)^{\frac{1}{p}} \left(\int_\Omega v^q dx \right)^{\frac{1}{q}}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \tag{2.1.7}$$

and the following well-known Sobolev interpolation results, e.g. see Theorem 3 in [1]: Let $p \in [1, \infty]$, $m \geq 1$ and $v \in W^{m,p}(\Omega)$. Then there are constants C and $\mu = \frac{d}{m} \left(\frac{1}{p} - \frac{1}{r} \right)$ such that the inequality

$$|v|_{0,r} \leq C |v|_{0,p}^{1-\mu} \|v\|_{m,p}^{\mu}, \quad \text{holds for } r \in \begin{cases} [p, \infty] & \text{if } m - \frac{d}{p} > 0, \\ [p, \infty) & \text{if } m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m - d/p}] & \text{if } m - \frac{d}{p} < 0. \end{cases} \tag{2.1.8}$$

We also state the following lemma, which will prove useful in our subsequent analysis.

Lemma 2.1.1 Let $u, v, \eta \in H^1(\Omega)$, $f = u - v$, $g = u^m v^{n-m}$, $m, n = 0, 1, 2$, and $n - m \geq 0$. Then for $d = 1, 2, 3$,

$$\left| \int_\Omega f g \eta dx \right| \leq C |u - v|_0 \, \|u\|_1^m \, \|v\|_1^{n-m} \, \|\eta\|_1. \tag{2.1.9}$$

Proof: Note that using the Cauchy-Schwarz inequality we have

$$|u^m v^{n-m}|_{0,p} \leq \begin{cases} |u|^m_{0,2mp}\ |v|^{(n-m)}_{0,2(n-m)p} & \text{for } n-m \neq 0, \text{ and } m \neq 0, \\ |u|^m_{0,mp} \text{ or } |v|^{(n-m)}_{0,(n-m)p} & \text{for } n-m = 0, \text{ or } m = 0 \text{ respectively.} \end{cases}$$

From the generalised Hölder inequality and the result above we have for $n = 2$,

$$\begin{aligned} \left| \int_\Omega fg\eta dx \right| &\leq |u-v|_0\ |u^m v^{n-m}|_{0,3}\ |\eta|_{0,6}, \\ &\leq |u-v|_0\ \ |\eta|_{0,6} \begin{cases} |u|^2_{0,6} & \text{for } m = 2, \\ |u|_{0,6}\ |v|_{0,6} & \text{for } m = 1, \\ |v|^2_{0,6} & \text{for } m = 0, \end{cases} \\ &\leq C|u-v|_0\ \|u\|_1^m\ \|v\|_1^{n-m}\ \|\eta\|_1, \end{aligned}$$

where we have noted (2.1.8) to obtain the last inequality. Similarly we can show for $n = 0, 1$. This ends the proof. □

2.2 The Existence and Uniqueness of the Continuous Problem

Given $\gamma > 0$ and $u_i^0 \in H^1(\Omega)$, for $i = 1, 2$, such that $\|u_1^0\|_1 + \|u_2^0\|_1 \leq C$, we consider the problem:

(P) Find $\{u_i, w_i\}$ such that $u_i \in H^1(0,T;(H^1(\Omega))') \cap L^\infty(0,T;H^1(\Omega))$ for *a.e.* $t \in (0,T)$, $w_i \in L^2(0,T;H^1(\Omega))$

$$\left\langle \frac{\partial u_1}{\partial t}, \eta \right\rangle = -(\nabla w_1, \nabla \eta), \tag{2.2.1a}$$

$$(w_1, \eta) = (\phi(u_1), \eta) + \gamma(\nabla u_1, \nabla \eta) + 2D(\Psi_1(u_1, u_2), \eta), \tag{2.2.1b}$$

$$u_1(x, 0) = u_1^0(x), \tag{2.2.1c}$$

and

$$\left\langle \frac{\partial u_2}{\partial t}, \eta \right\rangle = -(\nabla w_2, \nabla \eta), \tag{2.2.1d}$$

$$(w_2, \eta) = (\phi(u_2), \eta) + \gamma(\nabla u_2, \nabla \eta) + 2D(\Psi_2(u_1, u_2), \eta), \tag{2.2.1e}$$

$$u_2(x, 0) = u_2^0(x), \tag{2.2.1f}$$

for all $\eta \in H^1(\Omega)$ for *a.e.* $t \in (0, T)$, where $\phi(\cdot)$, $\Psi_1(\cdot, \cdot)$, and $\Psi_2(\cdot, \cdot)$ are given by (1.0.4g), (1.0.4h) and (1.0.4i) respectively.

Using (2.1.3), we can write (2.2.1a) and (2.2.1d) as

$$\big(\nabla(\mathcal{G}\frac{\partial u_i}{\partial t} + w_i), \nabla \eta\big) = 0 \quad \forall \eta \in H^1(\Omega). \tag{2.2.2}$$

Taking $\eta = \mathcal{G}\frac{\partial u_i}{\partial t} + w_i$ in (2.2.2), we have for *a.e.* $t \in (0, T)$

$$\left|\mathcal{G}\frac{\partial u_i}{\partial t} + w_i\right|_1^2 = 0,$$

which implies

$$\left|\mathcal{G}\frac{\partial u_i}{\partial t} + w_i\right|_1 = \left|\mathcal{G}\frac{\partial u_i}{\partial t} + w_i - \fint w_i\right|_1 = 0.$$

Thus by the Poincaré inequality (2.1.2) we have

$$0 = \left|\mathcal{G}\frac{\partial u_i}{\partial t} + w_i - \fint w_i\right|_1 \geq \widetilde{C}_P^{-1}\left|\mathcal{G}\frac{\partial u_i}{\partial t} + w_i - \fint w_i\right|_0.$$

Hence we obtain

$$w_i = -\mathcal{G}\frac{\partial u_i}{\partial t} + \fint w_i, \tag{2.2.3}$$

where

$$\fint w_i = \frac{1}{|\Omega|}\big((\phi(u_i), 1) + 2D(\Psi_i(u_1, u_2), 1)\big). \tag{2.2.4}$$

Noting (2.2.3), (2.2.4) and

$$(\varphi(r), \eta) - \frac{1}{|\Omega|}\big((\varphi(r), 1), \eta\big) = \big(\varphi(r), (I - \fint\ )\eta\big),$$

where $(I - \fint)\eta := \eta - \frac{1}{|\Omega|}(\eta, 1)$, we can restate the problem **(P)** as:
Find $\{u_1, u_2\}$ such that for $i = 1, 2$, $u_i \in H^1(0,T;(H^1(\Omega))') \cap L^\infty(0,T;H^1(\Omega))$, $u_i(\cdot,0) = u_i^0(\cdot)$ and for *a.e.* $t \in (0,T)$

$$(\mathcal{G}\frac{\partial u_1}{\partial t}, \eta) + \gamma(\nabla u_1, \nabla\eta) + \big(\phi(u_1) + 2D\Psi_1(u_1,u_2), (I - \fint)\eta\big) = 0, \tag{2.2.5a}$$

$$(\mathcal{G}\frac{\partial u_2}{\partial t}, \eta) + \gamma(\nabla u_2, \nabla\eta) + \big(\phi(u_2) + 2D\Psi_2(u_1,u_2), (I - \fint)\eta\big) = 0, \tag{2.2.5b}$$

for all $\eta \in H^1(\Omega)$.

Theorem 2.2.1 Given $u_i^0 \in H^1(\Omega)$, $i = 1, 2$, such that $\|u_1^0\|_1 + \|u_2^0\|_1 \leq C$ then there exists a unique solution $\{u_i, w_i\}$ to **(P)** such that the following stability bounds hold

$$\|u_i\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \tag{2.2.6a}$$

$$\|u_i\|_{H^1(0,T;(H^1(\Omega))')} \leq C, \tag{2.2.6b}$$

$$\|w_i\|_{L^2(0,T;H^1(\Omega))} \leq C(1 + T^{\frac{1}{2}}), \tag{2.2.6c}$$

where C is independent of T.

Proof. To prove the existence we use the Faedo-Galerkin method of Lions [27]. Let $\{z_j\}_{j=1}^\infty$ be the orthonormal basis for $H^1(\Omega)$ consisting of the eigenfunctions for

$$-\Delta z + z = \lambda z \quad \text{in } \Omega, \tag{2.2.7a}$$

$$\frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{2.2.7b}$$

Let V^k denote the finite dimensional subspace of $H^1(\Omega)$, spanned by $\{z_j\}_{j=1}^k$. Note that $z_1 = 1/|\Omega|^{\frac{1}{2}}$. The Galerkin approximation for the problem **(P)** is the following: Find $\{u_1^k, u_2^k, w_1^k, w_2^k\} \in V^k \times V^k \times V^k \times V^k$,

$$u_i^k = \sum_{j=1}^k c_{i,j}(t) z_j, \qquad w_i^k = \sum_{j=1}^k d_{i,j}(t) z_j, \quad \text{for } i = 1, 2 \tag{2.2.8a}$$

such that

$$\left(\frac{du_1^k}{dt}, \eta^k\right) = -(\nabla w_1^k, \nabla \eta^k) \quad \forall \eta^k \in V^k, \tag{2.2.8b}$$

$$(w_1^k, \eta^k) = (\phi(u_1^k), \eta^k) + \gamma(\nabla u_1^k, \nabla \eta^k) + 2D(\Psi_1(u_1^k, u_2^k), \eta^k) \quad \forall \eta^k \in V^k, \tag{2.2.8c}$$

$$u_1^k(x, 0) = P^k(u_1^0), \tag{2.2.8d}$$

and

$$\left(\frac{du_2^k}{dt}, \eta^k\right) = -(\nabla w_2^k, \nabla \eta^k) \quad \forall \eta^k \in V^k, \tag{2.2.8e}$$

$$(w_2^k, \eta^k) = (\phi(u_2^k), \eta^k) + \gamma(\nabla u_2^k, \nabla \eta^k) + 2D(\Psi_2(u_1^k, u_2^k), \eta^k) \quad \forall \eta^k \in V^k, \tag{2.2.8f}$$

$$u_2^k(x, 0) = P^k(u_2^0), \tag{2.2.8g}$$

where P^k is a projection from $H^1(\Omega)$ into V^k defined by

$$\left.\begin{array}{rcll} P^k v &=& \sum_{j=1}^k (v, z_j) z_j & \forall \eta^k \in V^k, \\ (P^k v - v, \eta^k) &=& (\nabla(P^k v - v), \nabla \eta^k) = 0 & \forall \eta^k \in V^k, \\ \|P^k\|_{\mathcal{L}(H^1, V^k)} &=& \|P^k\|_{\mathcal{L}(L^2, V^k)} = 1. & \end{array}\right\} \tag{2.2.9}$$

Straightforward calculation shows that this projection operator satisfies the following properties, for $i = 0, 1$,

$$|P^k v - v|_i \leq |\xi^k - v|_i \quad \forall \xi^k \in V^k, \tag{2.2.10}$$

$$|P^k v|_i \leq |v|_i \quad \forall v \in H^1(\Omega). \tag{2.2.11}$$

Since V^k is dense in $H^1(\Omega)$ and the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact (see Dautray and Lions [17] page 140) it follows that,

$$P^k v \to v \quad \text{strongly} \quad \text{in } L^2(\Omega). \tag{2.2.12}$$

Using (2.2.7a–b) and taking $\eta^k = z_j$ we can rewrite (2.2.8a-g) as a coupled system of first order differential equations, for $j = 1, 2, \ldots, k$,

$$\frac{dc^k_{1,j}(t)}{dt} = -(\lambda_j - 1)d^k_{1,j}(t), \tag{2.2.13a}$$

$$\begin{aligned} d^k_{1,j}(t) = {} & \big(f^k(\boldsymbol{c}^k_1(t))\big)_j - c^k_{1,j}(t) + \gamma(\lambda_j - 1)c^k_{1,j}(t) \\ & + 2D\big(g^k_1(\boldsymbol{c}^k_1(t), \boldsymbol{c}^k_2(t))\big)_j, \end{aligned} \tag{2.2.13b}$$

and

$$\frac{dc^k_{2,j}(t)}{dt} = -(\lambda_j - 1)d^k_{2,j}, \tag{2.2.13c}$$

$$\begin{aligned} d^k_{2,j}(t) = {} & \big(f^k(\boldsymbol{c}^k_2(t))\big)_j - c^k_{2,j}(t) + \gamma(\lambda_j - 1)c^k_{2,j}(t) \\ & + 2D\big(g^k_2(\boldsymbol{c}^k_1(t), \boldsymbol{c}^k_2(t))\big)_j, \end{aligned} \tag{2.2.13d}$$

where

$$\begin{aligned} \big(f^k(\boldsymbol{c}^k_i(t))\big)_j &:= \big((u^k_i)^3, z_j\big), \\ \big(g^k_1(\boldsymbol{c}^k_1(t), \boldsymbol{c}^k_2(t))\big)_j &:= \big((u^k_1 + 1)(u^k_2 + 1)^2, z_j\big), \\ \big(g^k_2(\boldsymbol{c}^k_1(t), \boldsymbol{c}^k_2(t))\big)_j &:= \big((u^k_2 + 1)(u^k_1 + 1)^2, z_j\big). \end{aligned}$$

The functions $f^k(\boldsymbol{c}^k_i(t))$, $g^k_1(\boldsymbol{c}^k_1(t), \boldsymbol{c}^k_2(t))$ and $g^k_2(\boldsymbol{c}^k_1(t), \boldsymbol{c}^k_2(t))$ are locally Lipschitz continuous functions of $\boldsymbol{c}^k_i$.

Letting $\boldsymbol{c}^k = [\boldsymbol{c}^k_1, \boldsymbol{c}^k_2]^T$ we can rewrite (2.2.13a–d) as $\frac{d\boldsymbol{c}^k}{dt} = \mathcal{H}(\boldsymbol{c}^k)$, which is locally Lipschitz continuous. Hence from the theory existence and uniqueness for systems of ordinary differential equations (see [8] for example) we deduce the local existence for u^k_i, w^k_i, $i = 1, 2$. To get existence of a global solution, we only need to obtain *a priori* estimates of u^k_i, w^k_i independent of k.

Now consider the free energy

$$\mathcal{E}(u_1, u_2) = \int_\Omega \left(\psi(u_1) + \frac{\gamma}{2}|\nabla u_1|^2 + \psi(u_2) + \frac{\gamma}{2}|\nabla u_2|^2 + 2D\Psi(u_1, u_2)\right)dx, \tag{2.2.14}$$

where $\psi(\cdot)$ and $\Psi(\cdot, \cdot)$ are given by (1.0.2b) and (1.0.2c) respectively.

Since $du^k_i(t)/dt \in V^k$ for $i = 1, 2$, differentiating $\mathcal{E}(u^k_1, u^k_2)$ with respect to t and rearranging the terms, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u_1^k(t),u_2^k(t)) = &\Big(\psi'(u_1^k(t)) + 2D\Psi_1(u_1^k,u_2^k), \frac{du_1^k(t)}{dt}\Big)\\ &+\gamma\Big(\nabla u_1^k(t), \nabla(\frac{du_1^k(t)}{dt})\Big)\\ &+\Big(\psi'(u_2^k(t)) + 2D\Psi_2(u_1^k,u_2^k), \frac{du_2^k(t)}{dt}\Big)\\ &+\gamma\Big(\nabla u_2^k(t), \nabla(\frac{du_2^k(t)}{dt})\Big),\end{aligned} \tag{2.2.15}$$

where $\Psi_1(\cdot,\cdot)$ and $\Psi_2(\cdot,\cdot)$ are given by (1.0.4h) and (1.0.4i) respectively. Noting (2.2.1b) and (2.2.1e), together with (2.2.1a) and (2.2.1d), and rearranging the terms we can express (2.2.15) as

$$\frac{d}{dt}\mathcal{E}(u_1^k(t),u_2^k(t)) + |w_1^k(t)|_1^2 + |w_2^k(t)|_1^2 = 0. \tag{2.2.16}$$

In particular

$$\frac{d}{dt}\mathcal{E}(u_1^k(t),u_2^k(t)) \leq 0, \tag{2.2.17}$$

i.e. $\mathcal{E}$ is a Lyapunov functional.

Integrating (2.2.16) over $(0,t)$ and rearranging the terms we have

$$\mathcal{E}(u_1^k(t),u_2^k(t)) + \int_0^t |w_1^k(s)|_1^2 ds + \int_0^t |w_2^k(s)|_1^2 ds \leq \mathcal{E}(P^k u_1^0, P^k u_2^0), \tag{2.2.18}$$

where we have noted (2.2.17), (2.2.8d) and (2.2.8g).

Note that $\psi(r) = \frac{1}{4}(r^2-1)^2$, so that

$$0 \leq \psi(r) \leq \frac{1}{4}r^4 + \frac{1}{4}.$$

Hence recalling (2.1.8) we have for $i = 1,2$,

$$\begin{aligned}\int_\Omega \psi(u_i^k(t))dx \leq &\frac{1}{4}\int_\Omega (u_i^k(t))^4 dx + \frac{1}{4}\int_\Omega dx\\ = &\frac{1}{4}|u_i^k(t)|_{0,4}^4 + \frac{1}{4}|\Omega| \leq C\|u_i^k(t)\|_1^4 + \frac{1}{4}|\Omega|.\end{aligned} \tag{2.2.19}$$

Noting the Cauchy-Schwarz inequality and the Young inequality (2.1.4) with $\epsilon = 1$

and $p = q = 2$ we have

$$
\begin{aligned}
\int_\Omega \Psi(r,s)dx &= \frac{1}{2}\int_\Omega (r+1)^2(s+1)^2 dx \\
&\le 2\int_\Omega \big(r^2s^2 + r^2 + s^2 + 1\big)dx \\
&\le 2\big(|r|_{0,4}^2\,|s|_{0,4}^2 + |r|_0^2 + |s|_0^2 + |\Omega|\big) \\
&\le |r|_{0,4}^4 + |s|_{0,4}^4 + 2|r|_0^2 + 2|s|_0^2 + 2|\Omega| \\
&\le C\|r\|_1^4 + C\|s\|_1^4 + 2|r|_0^2 + 2|s|_0^2 + 2|\Omega|,
\end{aligned} \tag{2.2.20}
$$

where we have noted (2.1.8) to obtain the last inequality.

By the strong convergence of $P^k u_i^0$ to u_i^0, $i = 1, 2$, in $L^2(\Omega)$, (2.2.19), (2.2.20), (2.2.11) and the assumption of the theorem, $\|u_1^0\|_1 + \|u_2^0\|_1 \le C$, we have

$$
\begin{aligned}
\mathcal{E}(P^k u_1^0, P^k u_2^0) &\le C\|P^k u_1^0\|_1^4 + C\|P^k u_2^0\|_1^4 + \big(4D + \frac{1}{2}\big)|\Omega| \\
&\quad + 4D|P^k u_1^0|_0^2 + 4D|P^k u_2^0|_0^2 + \frac{\gamma}{2}|P^k u_1^0|_1^2 + \frac{\gamma}{2}|P^k u_2^0|_1^2 \\
&\le C\|u_1^0\|_1^4 + C\|u_2^0\|_1^4 + \big(4D + \frac{1}{2}\big)|\Omega| \\
&\quad + 4D|u_1^0|_0^2 + 4D|u_2^0|_0^2 + \frac{\gamma}{2}|u_1^0|_1^2 + \frac{\gamma}{2}|u_2^0|_1^2 \le C,
\end{aligned} \tag{2.2.21}
$$

where C is independent of T and k. It follows from (2.2.18), and (2.2.21) that

$$
\frac{\gamma}{2}|u_1^k(t)|_1^2 + \int_0^t |w_1^k(s)|_1^2 ds + \frac{\gamma}{2}|u_2^k(t)|_1^2 + \int_0^t |w_2^k(s)|_1^2 ds \le C, \tag{2.2.22}
$$

where C is independent of T and k.

Now taking $\eta^k = 1$ in (2.2.8b) and (2.2.8e), we have for $i = 1, 2$, that

$$
\Big(\frac{du_i^k(t)}{dt}, 1\Big) = 0. \tag{2.2.23}
$$

Integrating both side of (2.2.23) over $(0, t)$, we obtain

$$
\begin{aligned}
0 &= \int_0^t \int_\Omega \frac{du_i^k}{ds} dx ds = \int_\Omega \int_0^t \frac{du_i^k(s)}{ds} ds dx = \int_\Omega \big(u_i^k(t) - u_i^k(0)\big)dx \\
&= (u_i^k(t), 1) - (u_i^k(0), 1).
\end{aligned}
$$

Hence

$$(u_i^k(t), 1) = (u_i^k(0), 1) = (P^k u_i^0, 1) = (u_i^0, 1), \tag{2.2.24}$$

which implies for any t that

$$|(u_i^k(t), 1)| \le C. \tag{2.2.25}$$

Using the Poincaré inequality (2.1.2), (2.2.22) and (2.2.25) we obtain

$$|u_i^k(t)|_0 \le \tilde{C}_P(|u_i^k(t)|_1 + |(u_i^k(t), 1)|) \le C. \tag{2.2.26}$$

The equations (2.2.25) and (2.2.26) imply that $u_i^k(t) \in H^1(\Omega)$, and it follows from (2.2.22) that

$$\|u_i^k\|_{L^\infty(0,T;H^1(\Omega))} \le C. \tag{2.2.27}$$

Recalling (2.1.3) and (2.2.3) we have for $i = 1, 2$,

$$\left\| \frac{du_i^k(t)}{dt} \right\|_{-1}^2 = \left| \mathcal{G} \frac{du_i^k(t)}{dt} \right|_1^2 = |w_i^k(t)|_1^2. \tag{2.2.28}$$

So setting $t = T$ we can rewrite (2.2.22) as

$$\frac{\gamma}{2}|u_1^k(T)|_1^2 + \int_0^T |w_1^k(t)|_1^2 dt + \frac{\gamma}{2}|u_2^k(T)|_1^2 + \int_0^T |w_2^k(t)|_1^2 dt \le C, \tag{2.2.29}$$

in particular

$$\int_0^T \left\| \frac{du_1^k(t)}{dt} \right\|_{-1}^2 dt + \int_0^T \left\| \frac{du_2^k(t)}{dt} \right\|_{-1}^2 dt \le C, \tag{2.2.30}$$

which implies for $i = 1, 2$, that

$$\left\| \frac{du_i^k(t)}{dt} \right\|_{L^2(0,T;(H^1(\Omega))')} \le C, \tag{2.2.31}$$

where C is independent of T and k.

To show that $u_i^k(t)$ is bounded in $L^2(0, T; (H^1(\Omega))')$, we show that $u_i^k(t)$ −

$\fint u_i^k(t) \in L^2(0,T;(H^1(\Omega))')$ since the mass is conserved. Noting (2.2.24) we have

$$\begin{aligned} u_i^k(t) - \fint u_i^k(t) &= u_i^k(t) - \frac{1}{|\Omega|}(u_i^k(t),1) = u_i^k(t) - \frac{1}{|\Omega|}(u_i^k(0),1) \\ &= u_i^k(t) - u_i^k(0) + u_i^k(0) - \frac{1}{|\Omega|}(u_i^k(0),1) \\ &= \int_0^t \frac{du_i^k}{ds} ds + u_i^k(0) - \frac{1}{|\Omega|}(u_i^k(0),1). \end{aligned}$$

Hence noting (2.1.6) and the Young inequality (2.1.4), setting $t = T$ in the integration on the right hand side, and using (2.2.31) we obtain

$$\begin{aligned} \left\| u_i^k(t) - \fint u_i^k(t) \right\|_{-1}^2 &\le \left(\left\| \int_0^T \frac{du_i^k}{ds} ds \right\|_{-1} + \left\| u_i^k(0) - \frac{1}{|\Omega|}(u_i^k(0),1) \right\|_{-1} \right)^2 \\ &\le \left(\left\| \int_0^T \frac{du_i^k}{ds} ds \right\|_{-1} + \widetilde{C}_P \left| u_i^k(0) - \frac{1}{|\Omega|}(u_i^k(0),1) \right|_0 \right)^2 \\ &\le \left(\left\| \int_0^T \frac{du_i^k}{ds} ds \right\|_{-1} + \widetilde{C}_P |u_i^k(0)|_0 + C|(u_i^k(0),1)|_0 \right)^2 \\ &\le C \left\| \int_0^T \frac{du_i^k}{ds} ds \right\|_{-1}^2 + C|u_i^k(0)|_0^2 + C|(u_i^k(0),1)|_0^2 \\ &\le C \int_0^T \left\| \frac{du_i^k}{ds} \right\|_{-1}^2 ds + C \le C. \end{aligned} \tag{2.2.32}$$

Integrating (2.2.32) over $(0,T)$ we obtain

$$\|u_i^k(t) - \fint u_i^k(t)\|_{L^2(0,T;(H^1(\Omega))')} \le C(T) \le C, \tag{2.2.33}$$

where we have noted to obtain the last inequality that by (2.2.17) for $T \to \infty$, $du_i^k/ds \to 0$.

Hence (2.2.31) and (2.2.33) imply that

$$\|u_1^k\|_{H^1(0,T;(H^1(\Omega))')} \le C. \tag{2.2.34}$$

Now we show $\|w_i^k\|_1$ is bounded. Setting $\xi = w_i^k$ in the Poincaré inequality (2.1.2) and noting the Young inequality (2.1.4) with $p = q = 2$, we have

$$|w_i^k|_0^2 \le C(|w_i^k|_1 + |(w_i^k,1)|)^2 \le C(|w_i^k|_1^2 + |(w_i^k,1)|^2), \tag{2.2.35}$$

Recalling

$$\|w_i^k\|_1^2 = |w_i^k|_0^2 + |w_i^k|_1^2, \tag{2.2.36}$$

and substituting (2.2.35) into (2.2.36), we have

$$\|w_i^k\|_1^2 \le C(|w_i^k|_1^2 + |(w_i^k, 1)|^2). \tag{2.2.37}$$

Thus by (2.2.22), it is enough to bound $|(w_i^k, 1)|$ to conclude $\|w_i^k\|_1$ is bounded.

Taking $\eta^k = 1$ in (2.2.8c) and (2.2.8f) we have for $i = 1, 2$,

$$(w_i^k(t), 1) = (\phi(u_i^k(t)), 1) + 2D(\Psi_i(u_1^k(t), u_1^k(t)), 1),$$

which implies that

$$|(w_i^k(t), 1)| \le |(\phi(u_i^k(t)), 1)| + 2D|(\Psi_i(u_1^k(t), u_1^k(t)), 1)|. \tag{2.2.38}$$

Noting the Young inequality (2.1.4), (2.1.8) and (2.2.27) we can bound the terms on the right hand side (2.2.38) as follows:

$$\begin{aligned} |(\phi(u_i^k(t)), 1)| &= \left| \int_\Omega \big((u_i^k(t))^3 - u_i^k(t)\big)dx \right| \\ &= \left| \int_\Omega \big((u_i^k(t))^2 - 1\big)u_i^k(t)dx \right| \\ &\le \frac{1}{2}\int_\Omega \big((u_i^k(t))^2 - 1\big)^2 dx + \frac{1}{2}\int_\Omega \big(u_i^k(t)\big)^2 dx \\ &\le \frac{1}{2}\int_\Omega \big((u_i^k(t))^4 + 1\big)dx \\ &\le \frac{1}{2}|u_i^k(t)|_{0,4}^4 + \frac{1}{2}|\Omega| \\ &\le \frac{1}{2}\|u_i^k(t)\|_1^4 + \frac{1}{2}|\Omega| \le C, \end{aligned} \tag{2.2.39}$$

and

$$
\begin{aligned}
|(\Psi_1(u_1^k(t), u_2^k(t)), 1)| &= \left| \int_\Omega \big((u_1^k(t) + 1)(u_2^k(t) + 1)^2\big) dx \right| \\
&\leq \int_\Omega \Big((u_1^k(t) + 1)^2 + \big((u_2^k(t))^2 + 1\big)^2\Big) dx \\
&\leq 2 \int_\Omega \big((u_1^k(t))^2 + (u_2^k(t))^4 + 2\big) dx \\
&= 2|u_1^k(t)|_0^2 + 2|u_2^k(t)|_{0,4}^4 + 4|\Omega| \\
&\leq 2|u_1^k(t)|_0^2 + C\|u_2^k(t)\|_1^4 + 4|\Omega| \\
&\leq C.
\end{aligned}
\tag{2.2.40}
$$

Similarly we can show that

$$
|(\Psi_2(u_1^k(t), u_2^k(t)), 1)| \leq C. \tag{2.2.41}
$$

Noting (2.2.39), (2.2.40) and (2.2.41), we conclude that

$$
|(w_i^k(t), 1)| \leq C. \tag{2.2.42}
$$

Substituting (2.2.42) into (2.2.37), integrating the resulting equation over $(0, T)$ and noting (2.2.29) we conclude that

$$
\|w_i^k(t)\|_{L^2(0,T;H^1(\Omega))} \leq C(1 + T^{\frac{1}{2}}),
$$

where C is independent of T and k.

Furthermore, since $L^\infty(0, T; H^1(\Omega)) \subset L^2(0, T; H^1(\Omega))$ we have

$$
\|u_i^k\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{2.2.43}
$$

Thus $u_i^k \in H^1(0, T; (H^1(\Omega))') \cap L^2(0, T; H^1(\Omega))$. Now since $H^1(0, T; (H^1(\Omega))')$ and $L^2(0, T; H^1(\Omega))$ are reflexive Banach spaces then by compactness arguments (see Dautray and Lions [17] page 289) we deduce the existence of subsequences $\{u_i^k, w_i^k\}$

such that

$$u_i^k \to u_i \quad \text{in} \quad H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^1(\Omega)) \quad \text{weakly}, \tag{2.2.44}$$

$$w_i^k \to w_i \quad \text{in} \quad L^2(0,T;H^1(\Omega)) \quad \text{weakly}. \tag{2.2.45}$$

Since $L^\infty(0,T;H^1(\Omega))$ is the dual of $L^1(0,T;(H^1(\Omega))')$ (see Renardy and Rogers [33] page 378), which is separable, we can extract a subsequence in $L^\infty(0,T;H^1(\Omega))$ such that

$$u_i^k \to u_i \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{weak-star}. \tag{2.2.46}$$

Note that $H^1(\Omega)$ and $(H^1(\Omega))'$ are reflexive, and the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact. Hence as a consequence of the compactness theorem of Lions (see Theorem 5.1 in Lions [27] page 56) we can extract a subsequence in $L^2(0,T,L^2(\Omega))$ such that

$$u_i^k \to u_i \quad \text{strongly} \quad \text{in} \quad L^2(0,T,L^2(\Omega)). \tag{2.2.47}$$

Moreover if $u_i \in L^2(0,T;H^1(\Omega))$ and $du_i/dt \in L^2(0,T;(H^1(\Omega))')$ then $u_i \in C(0,T;L^2(\Omega))$ *a.e.* (see Lemma 1.2 in Temam [34] page 261). This result together with (2.2.44) and the strong convergence of $P^k(u_i^0)$ to u_i^0 in $L^2(\Omega)$ implies that $u_i(0) = u_i^0$.

Now we will show that these limits satisfy the problem (**P**). For any $\eta \in H^1(\Omega)$ set $\eta^k = P^k\eta$ in (2.2.8b-c) and (2.2.8e-f), we have

$$\left(\frac{du_1^k}{dt}, P^k\eta\right) = -(\nabla w_1^k, \nabla P^k\eta) \quad \forall \eta \in H^1(\Omega), \tag{2.2.48a}$$

$$(w_1^k, P^k\eta) = (\phi(u_1^k), P^k\eta) + \gamma(\nabla u_1^k, \nabla P^k\eta) + 2D(\Psi_1(u_1^k, u_2^k), P^k\eta), \tag{2.2.48b}$$

and

$$\left(\frac{du_2^k}{dt}, P^k\eta\right) = -(\nabla w_2^k, \nabla P^k\eta) \quad \forall \eta \in H^1(\Omega), \tag{2.2.48c}$$

$$(w_2^k, P^k\eta) = (\phi(u_2^k), P^k\eta) + \gamma(\nabla u_2^k, \nabla P^k\eta) + 2D(\Psi_2(u_1^k, u_2^k), P^k\eta). \tag{2.2.48d}$$

Passing to the limit *a.e.* in (2.2.48a) and (2.2.48c) we have (2.2.1a) and (2.2.1d). To yield the results it remains to show that for $i = 1, 2$,

$$(\phi(u_i^k), P^k\eta) \to (\phi(u_i), \eta) \quad \text{as} \quad k \to \infty, \tag{2.2.49}$$

$$(\Psi_i(u_1^k, u_2^k), P^k\eta) \to (\Psi_i(u_1, u_2), \eta) \quad \text{as} \quad k \to \infty. \tag{2.2.50}$$

Recalling the Young inequality (2.1.4), (2.1.9), (2.2.12), and (2.2.44) we are able to show (2.2.49), that is

$$\begin{aligned}
&\big|(\phi(u_i^k), P^k\eta) - (\phi(u_i), \eta)\big| \\
&\quad \le \big|(\phi(u_i^k), P^k\eta - \eta)\big| + \big|(\phi(u_i^k) - \phi(u_i), \eta)\big| \\
&\quad = \big|(((u_i^k)^2 - 1)u_i^k, P^k\eta - \eta)\big| + \big|((u_i^k)^3 - (u_i)^3, \eta)\big| + \big|(u_i - u_i^k, \eta)\big| \\
&\quad = \big|(((u_i^k)^2 - 1)u_i^k, P^k\eta - \eta)\big| + \big|(u_i - u_i^k, \eta)\big| \\
&\qquad + \big|((u_i^k - u_i)((u_i^k)^2 + u_i u_i^k + (u_i)^2), \eta)\big| \\
&\quad \le C\Big((\|u_i^k\|_1^3 + |u_i^k|_0)|P^k\eta - \eta|_0 + |u_i^k|_0\, |u_i^k - u_i|_0 \\
&\qquad + |u_i^k - u_i|_0\, \|\eta\|_1 (\|u_i^k\|_1^2 + \|u_i^k\|_1\, \|u_i\|_1 + \|u_i\|_1^2)\Big) \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Now we write (2.2.50) for $i = 1$ as

$$\begin{aligned}
&|(\Psi_1(u_1^k, u_2^k), P^k\eta) - (\Psi_1(u_1, u_2), \eta)| \\
&\qquad \le |(\Psi_1(u_1^k, u_2^k) - \Psi_1(u_1^k, u_2), P^k\eta)| + |(\Psi_1(u_1^k, u_2), P^k\eta - \eta)| \\
&\qquad\quad + |(\Psi_1(u_1^k, u_2) - \Psi_1(u_1, u_2), \eta)| \\
&\qquad = I_1 + I_2 + I_3,
\end{aligned}$$

where I_j, $j = 1, 2, 3$ are the corresponding terms in the right hand side. We show in turn that each of these terms will tend to zero as $k \to \infty$.

Noting the Young inequality (2.1.4), (2.1.9), (2.2.12), and (2.2.44) we obtain

$$
\begin{aligned}
I_1 &= \big|\big((u_1^k+1)\big((u_2^k+1)^2-(u_2+1)^2\big), P^k\eta\big)\big| \\
&= \big|\big((u_1^k+1)(u_2^k+u_2+2)(u_2^k-u_2), P^k\eta\big)\big| \\
&= \big|\big(u_1^k u_2^k + u_1^k u_2 + 2u_1^k + u_2^k + u_2 + 2)(u_2^k-u_2), P^k\eta\big)\big| \\
&\le C\Big(|u_2^k-u_2|_0\ \|P^k\eta\|_1\big(\|u_1^k\|_1\ \|u_2^k\|_1 + \|u_1^k\|_1\ \|u_2\|_1 + \|u_1^k\|_1 + \|u_2^k\|_1 + \|u_2\|_1\big) \\
&\qquad + |u_2^k-u_2|_0\ |P^k\eta|_0\Big) \to 0 \quad \text{as} \quad k\to\infty,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \big|\big((u_1^k+1)(u_2+1)^2, P^k\eta-\eta\big)\big| \\
&\le 2\big|\big(|u_1^k+1|((u_2)^2+1), |P^k\eta-\eta|\big)\big| \\
&= \big|\big(u_1^k(u_2)^2+(u_2)^2+u_1^k+1), |P^k\eta-\eta|\big)\big| \\
&\le C|P^k\eta-\eta|_0\big(\|u_1^k\|_1\ \|u_2\|_1^2 + \|u_2\|_1^2 + \|u_1^k\|_1 + |\Omega|\big) \to 0 \quad \text{as} \quad k\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \big|\big((u_1^k-u_1)(u_2+1)^2, \eta\big)\big| \\
&\le 2\big(|u_1^k-u_1|((u_2)^2+1), |\eta|\big) \\
&\le C|u_1^k-u_1|_0\big(\|u_2\|_1^2\ \|\eta\|_1 + |\eta|_0\big) \to 0 \quad \text{as} \quad k\to\infty.
\end{aligned}
$$

Now we show that the limit is unique. Let $\{u_1^1, w_1^1, u_2^1, w_2^1\}$ and $\{u_1^2, w_1^2, u_2^2, w_2^2\}$ be two solutions of $(\mathbf{P})$. Define

$$
z_1^u = u_1^1 - u_1^2, \qquad z_2^u = u_2^1 - u_2^2. \tag{2.2.51}
$$

Substitute these solution into (2.2.5a–b) we have for $i=1,2$,

$$
(\mathcal{G}\frac{\partial u_i^1}{\partial t}, \eta) + \gamma(\nabla u_i^1, \nabla\eta) + (\phi(u_i^1), \eta) + 2D(\Psi_i(u_1^1, u_2^1), \eta) = 0, \tag{2.2.52}
$$

$$
(\mathcal{G}\frac{\partial u_i^2}{\partial t}, \eta) + \gamma(\nabla u_i^2, \nabla\eta) + (\phi(u_i^2), \eta) + 2D(\Psi_i(u_1^2, u_2^2), \eta) = 0. \tag{2.2.53}
$$

Subtracting (2.2.53) from (2.2.52) and summing for $i=1,2$, with $\eta = u_1^1 - u_1^2$ and

$\eta = u_2^1 - u_2^2$ respectively of resulting equations, we obtain

$$\begin{aligned}
(\mathcal{G}\frac{\partial z_1^u}{\partial t}, z_1^u) + (\mathcal{G}\frac{\partial z_2^u}{\partial t}, z_2^u) &+ \gamma(\nabla z_1^u, \nabla z_1^u) + \gamma(\nabla z_2^u, \nabla z_2^u) \\
&= \big(\phi(u_1^2) - \phi(u_1^1), u_1^1 - u_1^2\big) + \big(\phi(u_2^2) - \phi(u_2^1), u_2^1 - u_2^2\big) \\
&\quad + 2D\big(\Psi_1(u_1^2, u_2^2) - \Psi_1(u_1^1, u_2^1), u_1^1 - u_1^2\big) \\
&\quad + 2D\big(\Psi_2(u_1^2, u_2^2) - \Psi_2(u_1^1, u_2^1), u_2^1 - u_2^2\big),
\end{aligned} \tag{2.2.54}$$

where we have noted (2.2.51) for the terms on the left hand side. Note that from the convexity we have

$$r^3(s-r) \leq \frac{1}{4}s^4 - \frac{1}{4}r^4,$$

which implies

$$(r^3 - s^3)(r-s) \geq 0. \tag{2.2.55}$$

Hence by (2.2.55), and (2.1.5) we have

$$\begin{aligned}
\big(\phi(u_1^2) - \phi(u_1^1), u_1^1 - u_1^2\big) &+ \big(\phi(u_2^2) - \phi(u_2^1), u_2^1 - u_2^2\big) \\
&= \big((u_1^2)^3 - (u_1^1)^3 + u_1^1 - u_1^2, u_1^1 - u_1^2\big) \\
&\qquad + \big((u_2^2)^3 - (u_2^1)^3 + u_2^1 - u_2^2, u_2^1 - u_2^2\big) \\
&\leq (u_1^1 - u_1^2, u_1^1 - u_1^2) + (u_2^1 - u_2^2, u_2^1 - u_2^2) \\
&= (z_1^u, z_1^u) + (z_2^u, z_2^u) \\
&= (\nabla z_1^u, \nabla \mathcal{G} z_1^u) + (\nabla z_2^u, \nabla \mathcal{G} z_2^u) \\
&\leq |z_1^u|_1 \, \|z_1^u\|_{-1} + |z_2^u|_1 \, \|z_2^u\|_{-1} \\
&\leq \frac{\alpha}{2}(|z_1^u|_1^2 + |z_2^u|_1^2) + \frac{1}{2\alpha}(\|z_1^u\|_{-1}^2 + \|z_2^u\|_{-1}^2).
\end{aligned} \tag{2.2.56}$$

Now the Taylor expansion of Ψ about (u_1^2, u_2^2) and (u_1^1, u_2^1) are respectively given by

$$\begin{aligned}\Psi(u_1^1,u_2^1) = {}& \Psi(u_1^2,u_2^2) + \frac{\partial\Psi(u_1^2,u_2^2)}{\partial u_1}(u_1^1-u_1^2) + \frac{\partial\Psi(u_1^2,u_2^2)}{\partial u_2}(u_2^1-u_2^2)\\ &+ \frac{1}{2}\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1^{(2)}}(u_1^1-u_1^2)^2 + \frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2}(u_1^1-u_1^2)(u_2^1-u_2^2)\\ &+ \frac{1}{2}\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_2^{(2)}}(u_2^1-u_2^2)^2, \end{aligned} \tag{2.2.57a}$$

and

$$\begin{aligned}\Psi(u_1^2,u_2^2) = {}& \Psi(u_1^1,u_2^1) + \frac{\partial\Psi(u_1^1,u_2^1)}{\partial u_1}(u_1^2-u_1^1) + \frac{\partial\Psi(u_1^1,u_2^1)}{\partial u_2}(u_2^2-u_2^1)\\ &+ \frac{1}{2}\frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1^{(2)}}(u_1^2-u_1^1)^2 + \frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}(u_1^2-u_1^1)(u_2^2-u_2^1)\\ &+ \frac{1}{2}\frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_2^{(2)}}(u_2^2-u_2^1)^2, \end{aligned} \tag{2.2.57b}$$

where ζ_1 and ξ_1 are between u_1^1 and u_1^2, and ζ_2 and ξ_2 are between u_2^1 and u_2^2.

Adding (2.2.57a) and (2.2.57b), simplifying, integrating over Ω, and noting (1.0.4h) and (1.0.4i) we obtain

$$\begin{aligned} 0 = {}& \int_\Omega (\Psi_1(u_1^2,u_2^2) - \Psi_1(u_1^1,u_2^1))(u_1^1-u_1^2)dx\\ &+ \int_\Omega (\Psi_2(u_1^2,u_2^2) - \Psi_2(u_1^1,u_2^1))(u_2^1-u_2^2)dx\\ &+ \frac{1}{2}\int_\Omega \left(\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1^{(2)}} + \frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1^{(2)}}\right)(u_1^1-u_1^2)^2dx\\ &+ \frac{1}{2}\int_\Omega \left(\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_2^{(2)}} + \frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_2^{(2)}}\right)(u_2^1-u_2^2)^2dx\\ &+ \int_\Omega \left(\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2} + \frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}\right)(u_1^1-u_1^2)(u_2^1-u_2^2)dx, \end{aligned}$$

so that using the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\left((\Psi_1(u_1^2,u_2^2)-\Psi_1(u_1^1,u_2^1)),(u_1^1-u_1^2)\right)+\left((\Psi_2(u_1^2,u_2^2)-\Psi_2(u_1^1,u_2^1)),(u_2^1-u_2^2)\right)\right|\\
&\quad\le\frac{1}{2}\int_\Omega\left(\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1^{(2)}}+\frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1^{(2)}}\right)(u_1^1-u_1^2)^2dx\\
&\qquad+\frac{1}{2}\int_\Omega\left(\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_2^{(2)}}+\frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_2^{(2)}}\right)(u_2^1-u_2^2)^2dx\\
&\qquad+\frac{1}{2}\int_\Omega\left(\left|\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2}\right|+\left|\frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}\right|\right)(u_1^1-u_1^2)^2dx\\
&\qquad+\frac{1}{2}\int_\Omega\left(\left|\frac{\partial^{(2)}\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2}\right|+\left|\frac{\partial^{(2)}\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}\right|\right)(u_2^1-u_2^2)^2dx. \qquad (2.2.58)
\end{aligned}$$

Noting the Young inequality (2.1.4) with $\epsilon=1$, $p=q=2$, and for any s_1^1,s_1^2 between u_1^1 and u_1^2 and any s_2^1,s_2^2 between u_2^1 and u_2^2 we have

$$\frac{\partial^{(2)}\Psi(s_1^1,s_2^1)}{\partial u_1^{(2)}}=(s_2^1+1)^2\le 2((s_2^1)^2+1)\le 2\big((u_2^1)^2+(u_2^2)^2+1\big), \qquad (2.2.59a)$$

$$\frac{\partial^{(2)}\Psi(s_1^2,s_2^2)}{\partial u_1^{(2)}}=(s_2^2+1)^2\le 2\big((u_2^1)^2+(u_2^2)^2+1\big), \qquad (2.2.59b)$$

$$\frac{\partial^{(2)}\Psi(s_1^1,s_2^1)}{\partial u_2^{(2)}}=(s_1^1+1)^2\le 2\big((u_1^1)^2+(u_1^2)^2+1\big), \qquad (2.2.59c)$$

$$\frac{\partial^{(2)}\Psi(s_1^2,s_2^2)}{\partial u_2^{(2)}}=(s_1^2+1)^2\le 2\big((u_1^1)^2+(u_1^2)^2+1\big), \qquad (2.2.59d)$$

$$\begin{aligned}
\left|\frac{\partial^{(2)}\Psi(s_1^1,s_2^1)}{\partial u_1\partial u_2}\right|&=2|s_1^1+1|\,|s_2^1+1|\le(s_1^1+1)^2+(s_2^1+1)^2\\
&\le 2\big((u_1^1)^2+(u_1^2)^2+(u_2^1)^2+(u_2^2)^2+2\big), \qquad (2.2.59e)
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial^{(2)}\Psi(s_1^2,s_2^2)}{\partial u_1\partial u_2}\right|&=2|s_1^2+1|\,|s_2^2+1|\le(s_1^2+1)^2+(s_2^2+1)^2\\
&\le 2\big((u_1^1)^2+(u_1^2)^2+(u_2^1)^2+(u_2^2)^2+2\big). \qquad (2.2.59f)
\end{aligned}$$

Hence we can rewrite (2.2.58) as

$$\begin{aligned}
&\big|\big(\Psi_1(u_1^2,u_2^2)-\Psi_1(u_1^1,u_2^1),(u_1^1-u_1^2)\big)+\big(\Psi_2(u_1^2,u_2^2)-\Psi_2(u_1^1,u_2^1),(u_2^1-u_2^2)\big)\big|\\
&\quad\le 2\int_\Omega\big((u_2^1)^2+(u_2^2)^2+1\big)(u_1^1-u_1^2)^2dx\\
&\qquad+2\int_\Omega\big((u_1^1)^2+(u_1^2)^2+1\big)(u_2^1-u_2^2)^2dx\\
&\qquad+2\int_\Omega\big((u_1^1)^2+(u_1^2)^2+(u_2^1)^2+(u_2^2)^2+2\big)(u_1^1-u_1^2)^2dx\\
&\qquad+2\int_\Omega\big((u_1^1)^2+(u_1^2)^2+(u_2^1)^2+(u_2^2)^2+2\big)(u_2^1-u_2^2)^2dx. \qquad (2.2.60)
\end{aligned}$$

On noting (2.1.5) we obtain

$$\int_\Omega (u_i^1-u_i^2)^2dx=|z_i^u|_{0,2}^2\le(\nabla z_i^u,\nabla\mathcal{G}z_i^u)\le\frac{\alpha}{2}|z_i^u|_1^2+\frac{1}{2\alpha}\|z_i^u\|_{-1}^2. \qquad (2.2.61)$$

The Hölder inequality (2.1.7), (2.1.8), (2.2.27), (2.1.5), the Poincaré inequality (2.1.2) and the Young inequality (2.1.4) with $p=8/7,\ q=8$ yield, for $i,j,k=1,2,$

$$\begin{aligned}
\int_\Omega (u_k^j)^2(u_i^1-u_i^2)^2dx&\le\Big(\int_\Omega(u_k^j)^4dx\Big)^{\frac12}\Big(\int_\Omega(u_i^1-u_i^2)^4dx\Big)^{\frac12}\\
&=|u_k^j|_{0,4}^2\,|u_i^1-u_i^2|_{0,4}^2\le C\|u_k^j\|_1^2\,|u_i^1-u_i^2|_0^{\frac12}\,\|u_i^1-u_i^2\|_1^{\frac32},\\
&\le C|z_i^u|_0^{\frac12}\,\|z_i^u\|_1^{\frac32}=C(\nabla z_i^u,\nabla\mathcal{G}z_i^u)^{\frac14}\|z_i^u\|_1^{\frac32},\\
&\le C|z_i^u|_1^{\frac14}\,\|z_i^u\|_{-1}^{\frac14}\,\|z_i^u\|_1^{\frac32}\le C|z_i^u|_1^{\frac74}\,\|z_i^u\|_{-1}^{\frac14},\\
&\le\frac{7\epsilon}{8}|z_i^u|_1^2+C(\epsilon^{-1})\|z_i^u\|_{-1}^2. \qquad (2.2.62)
\end{aligned}$$

Noting

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z_i^u\|_{-1}^2=\frac{\mathrm{d}}{dt}\int_\Omega|\nabla\mathcal{G}z_i^u|^2dx=2\int_\Omega(\nabla\mathcal{G}z_i^u,\nabla\mathcal{G}\frac{\partial z_i^u}{\partial t})=2(\mathcal{G}\frac{\partial z_i^u}{\partial t},z_i^u), \qquad (2.2.63)$$

(2.2.56), and (2.2.60–2.2.63), we can rewrite (2.2.54) as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|z_1^u\|_{-1}^2+\|z_2^u\|_{-1}^2\big)+2\gamma\big(|z_1^u|_1^2+|z_2^u|_1^2\big)\\
&\qquad\le\big((12D+1)\alpha+42D\epsilon\big)\big(|z_1^u|_1^2+|z_2^u|_1^2\big)+C\big(\|z_1^u\|_{-1}^2+\|z_2^u\|_{-1}^2\big).
\end{aligned}$$

Setting $\alpha = \epsilon$ and $\epsilon = 7\gamma/(224D + 4), D \geq 0$, and rearranging the terms we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|z_1^u\|_{-1}^2 + \|z_2^u\|_{-1}^2\right) + \frac{\gamma}{4}\left(|z_1^u|_1^2 + |z_2^u|_1^2\right) \leq C\left(\|z_1^u\|_{-1}^2 + \|z_2^u\|_{-1}^2\right). \tag{2.2.64}$$

Integrating over $t \in (0, T)$ and using a Grönwall inequality, we conclude from (2.2.64) that

$$\|z_1^u(t)\|_{-1}^2 + \|z_2^u(t)\|_{-1}^2 + \frac{\gamma}{4}\int_0^t \left(|z_1^u(s)|_1^2 + |z_2^u(s)|_1^2\right) ds \leq \|z_1^u(0)\|_{-1}^2 + \|z_2^u(0)\|_{-1}^2 = 0.$$

Noting the Poincaré inequality (2.1.2), $(z_i^u, 1) = 0$, we obtain the uniqueness of u_i. The uniqueness of w_i follows from (2.2.3) and (2.2.4). This ends the proof of the existence and uniqueness of the problem (**P**). □

Below, we shall discuss a regularity result that will be used later in our subsequent error analysis.

2.3 Regularity

We suppose $\partial\Omega$ to be sufficiently smooth so that if z is a weak solution of

$$-\Delta z + z = f \quad \text{in } \Omega, \tag{2.3.1a}$$

$$\frac{\partial z}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{2.3.1b}$$

where $f \in \mathcal{F} \cap L^2(\Omega)$ then

$$\|z\|_2 \leq C|-\Delta z + z|_0, \tag{2.3.1c}$$

see Theorem (3.1.2.3) in [22] for a convex domain or a smooth boundary for example.

Proposition 2.3.1 For Ω sufficiently smooth, we have the following regularity results:

$$u_i \in L^2(0,T;H^2(\Omega)), \quad \text{and} \quad \frac{\partial u_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \quad \text{for } a.e.\ t. \tag{2.3.2}$$

Proof. Noting (2.3.1c) and (2.2.27) we have

$$\|u_i^k\|_2 \le C|-\Delta u_i^k + u_i^k|_0 \le C\big(|\Delta u_i^k|_0 + |u_i^k|_0\big) \le C|\Delta u_i^k|_0. \tag{2.3.3}$$

Squaring both sides of (2.3.3), summing the resulting equation for $i = 1,2$, and integrating over $t \in [0,T]$ we have

$$\int_0^t \big(\|u_1^k\|_2^2 + \|u_2^k\|_2^2\big) ds \le C\int_0^t \big(|\Delta u_1^k|_0 + |\Delta u_2^k|_0\big) ds. \tag{2.3.4}$$

Setting $\eta^k = -\Delta u_1^k \in V^k$ in (2.2.8c) and noting $\Delta = -(\nabla)^2$ we obtain

$$(\nabla w_1^k, \nabla u_1^k) = (\nabla\phi(u_1^k), \nabla u_1^k) + \gamma(\Delta u_1^k, \Delta u_1^k) + 2D(\nabla\Psi_1(u_1^k, u_2^k), \nabla u_1^k).$$

Noting (2.2.8b) and

$$(\frac{\partial u_1^k}{\partial t}, u_1^k) = \frac{1}{2}\frac{\partial}{\partial t}|u_1^k|_0^2,$$

we have

$$\gamma|\Delta u_1^k|_0^2 + \frac{1}{2}\frac{\partial}{\partial t}|u_1^k|_0^2 + (\nabla\phi(u_1^k), \nabla u_1^k) + 2D(\nabla\Psi_1(u_1^k, u_2^k), \nabla u_1^k) = 0. \tag{2.3.5}$$

Since

$$\begin{aligned}(\nabla\phi(u_1^k), \nabla u_1^k) &= (\nabla(u_1^k)^3 - \nabla u_1^k, \nabla u_1^k), \\ &= 3((u_1^k)^2\nabla u_1^k, \nabla u_1^k) - |u_1^k|_1^2, \\ &= 3|u_1^k \nabla u_1^k|_0^2 - |u_1^k|_1^2,\end{aligned}$$

and

$$\begin{aligned}(\nabla\Psi_1(u_1^k,u_2^k),\nabla u_1^k) &= \big(\nabla\big((u_1^k+1)(u_2^k+1)^2\big),\nabla u_1^k\big),\\ &= 2((u_1^k+1)(u_2^k+1)\nabla u_2^k,\nabla u_1^k) + ((u_2^k+1)^2\nabla u_1^k,\nabla u_1^k),\\ &= 2((u_1^k+1)\nabla u_2^k,(u_2^k+1)\nabla u_1^k) + |(u_2^k+1)\nabla u_1^k|_0^2,\end{aligned}$$

we can rewrite (2.3.5) as

$$\begin{aligned}\gamma|\Delta u_1^k|_0^2 + \frac{1}{2}\frac{\partial}{\partial t}|u_1^k|_0^2 + 3|u_1^k\nabla u_1^k|_0^2 + 2D|(u_2^k+1)\nabla u_1^k|_0^2\\ = |u_1^k|_1^2 - 4D\big((u_1^k+1)\nabla u_2^k,(u_2^k+1)\nabla u_1^k\big).\end{aligned} \tag{2.3.6}$$

Similarly we have

$$\begin{aligned}\gamma|\Delta u_2^k|_0^2 + \frac{1}{2}\frac{\partial}{\partial t}|u_2^k|_0^2 + 3|u_2^k\nabla u_2^k|_0^2 + 2D|(u_1^k+1)\nabla u_2^k|_0^2\\ = |u_2^k|_1^2 - 4D\big((u_2^k+1)\nabla u_1^k,(u_1^k+1)\nabla u_2^k\big).\end{aligned} \tag{2.3.7}$$

Adding (2.3.6) and (2.3.7), and noting the Cauchy-Schwarz inequality we have

$$\begin{aligned}\gamma(|\Delta u_1^k|_0^2 + |\Delta u_2^k|_0^2) + \frac{1}{2}\frac{\partial}{\partial t}(|u_1^k|_0^2 + |u_2^k|_0^2) + 3(|u_1^k\nabla u_1^k|_0^2 + |u_2^k\nabla u_2^k|_0^2)\\ + 2D(|(u_2^k+1)\nabla u_1^k|_0^2 + |(u_1^k+1)\nabla u_2^k|_0^2),\\ = |u_1^k|_1^2 + |u_2^k|_1^2 - 8D\big((u_1^k+1)\nabla u_2^k,(u_2^k+1)\nabla u_1^k\big),\\ \le |u_1^k|_1^2 + |u_2^k|_1^2 + 4D\big|\big((u_1^k+1)\nabla u_2^k,(u_2^k+1)\nabla u_1^k\big)\big|\\ + 4D\big|\big((u_2^k+1)\nabla u_2^k,(u_1^k+1)\nabla u_1^k\big)\big|.\end{aligned} \tag{2.3.8}$$

Note that

$$\big|\big((u_1^k+1)\nabla u_2^k,(u_2^k+1)\nabla u_1^k\big)\big| \le \frac{1}{2}\big(|(u_1^k+1)\nabla u_2^k|_0^2 + |(u_2^k+1)\nabla u_1^k|_0^2\big),$$

and

$$\left|\left((u_2^k+1)\nabla u_2^k,(u_1^k+1)\nabla u_1^k\right)\right| \le \frac{1}{2}|(u_2^k+1)\nabla u_2^k|_0^2 + \frac{1}{2}|(u_1^k+1)\nabla u_1^k|_0^2,$$
$$\le |u_2^k\nabla u_2^k|_0^2 + |\nabla u_2^k|_0^2 + |u_1^k\nabla u_1^k|_0^2 + |\nabla u_1^k|_0^2.$$

Thus we can rewrite (2.3.8) as

$$\gamma(|\Delta u_1^k|_0^2 + |\Delta u_2^k|_0^2) + \frac{1}{2}\frac{\partial}{\partial t}(|u_1^k|_0^2 + |u_2^k|_0^2)$$
$$\le 4D\big(|u_1^k\nabla u_1^k|_0^2 + |u_2^k\nabla u_2^k|_0^2\big) + (1+4D)|u_1^k|_1^2 + |u_2^k|_1^2. \tag{2.3.9}$$

Noting the Cauchy-Schwarz inequality, (2.1.8), the Poincaré inequality (2.1.2), (2.2.24), and the Young inequality (2.1.4) with $p = 4/d$, $q = 4/(4-d)$, we obtain

$$|u_i^k\nabla u_i^k|_0^2 \le |u_i^k|_{0,4}^2\ |u_i^k|_{1,4}^2,$$
$$\le C|u_i^k|_0^{2-\frac{d}{2}}\ \|u_i^k\|_1^{\frac{d}{2}}\ |u_i^k|_1^{2-\frac{d}{2}}\ \|u_i^k\|_2^{\frac{d}{2}},$$
$$\le C|u_i^k|_0^{2-\frac{d}{2}}\ |u_i^k|_1^2\ \|u_i^k\|_2^{\frac{d}{2}},$$
$$\le \frac{d\epsilon}{4}\|u_i^k\|_2^2 + C(\epsilon^{-1})|u_i^k|_0^2\ \ |u_i^k|_1^{\frac{8}{4-d}}. \tag{2.3.10}$$

Thus substituting (2.3.10) into (2.3.9), integrating the resulting equation over $t\in[0,T]$ and noting (2.2.27) we have

$$\frac{1}{2\gamma}\big(|u_1^k(t)|_0^2 + |u_2^k(t)|_0^2\big) + \int_0^t (|\Delta u_1^k|_0^2 + |\Delta u_2^k|_0^2)ds$$
$$\le \frac{Dd\epsilon}{\gamma}\int_0^t \big(\|u_1^k\|_2^2 + \|u_2^k\|_2^2\big)ds + C\int_0^t \big(|u_1^k|_1^2 + |u_2^k|_1^2\big)ds$$
$$+ C\int_0^t \big(|u_1^k|_0^2\ \ |u_1^k|_1^{\frac{8}{4-d}} + |u_2^k|_0^2\ \ |u_2^k|_1^{\frac{8}{4-d}}\big)ds + \frac{1}{2\gamma}\big(|u_1^k(0)|_0^2 + |u_2^k(0)|_0^2\big),$$

or

$$\int_0^t (|\Delta u_1^k|_0^2 + |\Delta u_2^k|_0^2) ds \leq \frac{Dd\epsilon}{\gamma} \int_0^t (\|u_1^k\|_2^2 + \|u_2^k\|_2^2) ds + C \int_0^t (|u_1^k|_1^2 + |u_2^k|_1^2) ds$$
$$+ C \int_0^t \left(|u_1^k|_0^2 \; |u_1^k|_1^{\frac{8}{4-d}} + |u_2^k|_0^2 \; |u_2^k|_1^{\frac{8}{4-d}} \right) ds$$
$$+ \frac{1}{2\gamma} \left(|u_1^k(0)|_0^2 + |u_2^k(0)|_0^2 \right). \tag{2.3.11}$$

Substituting (2.3.11) into (2.3.4) and choosing $\epsilon = \gamma^2/4DdC$ we obtain

$$\frac{3\gamma}{4} \int_0^t (\|u_1^k\|_2^2 + \|u_2^k\|_2^2) ds \leq C \Bigg(\int_0^t (|u_1^k|_1^2 + |u_2^k|_1^2) ds$$
$$+ \int_0^t \left(|u_1^k|_0^2 \; |u_1^k|_1^{\frac{8}{4-d}} + |u_2^k|_0^2 \; |u_2^k|_1^{\frac{8}{4-d}} \right) ds$$
$$+ \left(|u_1^k(0)|_0^2 + |u_2^k(0)|_0^2 \right) \Bigg). \tag{2.3.12}$$

It follows from (2.2.6a) and (2.3.12) that

$$\|u_i^k\|_{L^2(0,T;H^2(\Omega))} \leq C,$$

which is independent of k.

Since $L^2(0,T;H^2(\Omega))$ is a reflexive Banach space (see Ženišek [37] page 40) then by compactness arguments (see Dautray and Lions [17] page 289), we deduce the existence of subsequences $\{u_i^k\} \in L^2(0,T;H^2(\Omega))$ such that

$$u_i^k \to u_i \quad \text{in} \quad L^2(0,T;H^2(\Omega)) \quad \text{weakly.}$$

Thus $u_i \in L^2(0,T;H^2(\Omega))$. Furthermore since $\partial u_i^k/\partial \nu = 0$ on $\partial\Omega$, it follows by the weak convergence of $u_i^k \to u_i$ in $H^2(\Omega)$ that $\partial u_i/\partial \nu = 0$ on $L^2(\partial\Omega)$. □

Chapter 3

A Semidiscrete Approximation

In this chapter we introduce some notation which will be used in the current and following chapters. For completeness, we prove interpolation error estimates in the finite element space as these are necessary tools for analysis in the current chapter and chapter 4. Then a semidiscrete finite element approximation is proposed where the existence and uniqueness are proven. An error bound between the semidiscrete and continuous solution is given is the final section.

3.1 Notation

We shall now describe a semidiscrete approximation of the weak formulation of (1.0.7a-1.0.7f). We will assume the following:

(A) Let $\mathcal{T}^h$ be a quasi-uniform partition of $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, into disjoint simplices , τ, with $h_\tau = diam(\tau)$ and $h = \max_{\tau \in \mathcal{T}^h} h_\tau$, so that $\bar{\Omega} = \cup_{\tau \in \mathcal{T}^h} \bar{\tau}$, see Ciarlet [13] page 132. In addition, it is assumed that $\mathcal{T}^h$ is an acute partition; that is for (i) $d = 2$, the angle of any triangle does not exceed $\pi/2$. In fact this case can be relaxed to weakly acute, see Nochetto [29]; that is the sum of opposite angles relative to any side does not exceed π. (ii) $d = 3$ the angle between any two faces of the same tetrahedron does not exceed $\pi/2$.

Let $S^h \subset H^1(\Omega)$ be a finite element space defined by

$$S^h := \{\chi \in C(\bar{\Omega}) : \chi|_\tau \text{ is linear } \forall \tau \in \mathcal{T}^h\}.$$

Denote by $\{x_i\}_{i=1}^J$ the set of nodes of $\mathcal{T}^h$ and let $\{\eta_i\}_{i=1}^J$ be a basis for S^h defined by $\eta_i(x_j) = \delta_{ij}$, for $i, j = 1, \ldots, J$.

Let $\pi^h : C(\bar{\Omega}) \mapsto S^h$ be the interpolation operator such that $\pi^h\chi(x_i) = \chi(x_i)$, for $i = 1, \ldots, J$ and define a discrete inner product on $C(\bar{\Omega})$ as follows

$$(\chi_1, \chi_2)^h := \int_\Omega \pi^h(\chi_1(x)\chi_2(x))dx \equiv \sum_{i=1}^J m_i\chi_1(x_i)\chi_2(x_i), \tag{3.1.1}$$

where $m_i = (\eta_i, \eta_i)^h$. The induced norm $\|\cdot\|_h := [(\cdot,\cdot)^h]^{\frac{1}{2}}$ on S^h is equivalent to $|\cdot|_0 := [(\cdot,\cdot)]^{\frac{1}{2}}$. Note that the integral (3.1.1) can easily be computed by means of vertex quadrature rule, which is exact for piecewise linear functions (see Ciarlet [13] page 182).

Below we recall some well-known results about S^h (see [32], [15] respectively)

$$C_1|\chi|_0 \leq |\chi|_h \leq C_2|\chi|_0 \quad \forall \chi \in S^h, \tag{3.1.2a}$$

$$|(\eta, \chi) - (\eta, \chi)^h| \leq Ch^{1+r}\|\eta\|_1\|\chi\|_r \quad \forall \chi, \eta \in S^h, \quad r = 0, 1. \tag{3.1.2b}$$

We also note the following the interpolation error in $H^2(\Omega)$ (see Theorem 3.1.6 in Ciarlet [13]),

$$|(I - \pi^h)\eta|_m \leq Ch^{2-m}|\eta|_2 \qquad \forall \eta \in H^2(\Omega), \quad m = 0 \text{ or } 1. \tag{3.1.2c}$$

The Poincaré inequality (2.1.2) together with (3.1.2a) and (3.1.2b) yields the discrete Poincaré inequality, for h sufficiently small,

$$\left((\xi, \xi)^h\right)^{\frac{1}{2}} =: |\xi|_h \leq C_P(|\xi|_1 + |(\xi, 1)^h|), \tag{3.1.3}$$

where C_P is a constant independent of h.

Similar to (2.1.1) we introduce the discrete Green's operator $\mathcal{G}^h : \mathcal{F} \mapsto V^h$ such that

$$(\nabla\mathcal{G}^h v, \nabla\eta) = \langle v, \eta\rangle \quad \forall \eta \in S^h. \tag{3.1.4}$$

where $V^h := \{\eta^h \in S^h : (\eta^h, 1) = 0\}$. We define a norm on $\mathcal{F}$ as

$$\|\eta\|_{-h}^2 := |\mathcal{G}^h\eta|_1^2 = \langle \eta, \mathcal{G}^h\eta\rangle = \langle \mathcal{G}^h\eta, \eta\rangle \quad \forall \eta \in \mathcal{F}. \tag{3.1.5}$$

We have the following analogue of (2.1.5) and (2.1.6) respectively, that is for all $\alpha > 0$,

$$|v^h|_0^2 \equiv (\nabla \mathcal{G}^h v^h, \nabla v^h) \leq \|v^h\|_{-h}|v^h|_1 \leq \frac{\alpha}{2}|v^h|_1^2 + \frac{1}{2\alpha}\|v^h\|_{-h}^2 \quad \forall v^h \in V^h. \tag{3.1.6}$$

and

$$\|v^h\|_{-h}^2 \leq C_P|\mathcal{G}^h v^h|_1|v^h|_0. \tag{3.1.7}$$

We note that $\mathcal{G}^h$ satisfies the error estimate (see Nochetto [29] page 49)

$$|(\mathcal{G} - \mathcal{G}^h)\eta|_0 \leq Ch^{2-m}\|\eta\|_{-m} \quad \forall \eta \in (H^m(\Omega))' \cap \mathcal{F}, \quad m = 0, 1. \tag{3.1.8}$$

For later purposes, we recall the inverse inequality for $1 \leq p_1 \leq p_2 \leq \infty$ and $m = 0$ or 1 (see Ciarlet [13] page 140),

$$|\chi|_{m,p_2} \leq Ch^{\frac{d(p_1-p_2)}{p_1p_2}}|\chi|_{m,p_1} \quad \forall \chi \in S^h, \tag{3.1.9}$$

and the Sobolev embedding, for $d = 1, 2$, (see Lemma (5.4) in Thomée [35] for $d = 2$),

$$\|\chi\|_{0,\infty} \leq C\Big(\ln\frac{1}{h}\Big)^{\frac{d-1}{2}}|\chi|_1 \quad \forall \chi \in S^h. \tag{3.1.10}$$

We also note the following inequalities (see Barrett and Blowey [2])

$$C_1h^2|v^h|_1 \leq C_2h|v^h|_0 \leq \|v^h\|_{-h} \leq \|v^h\|_{-1} \leq C_3\|v^h\|_{-h} \quad \forall v^h \in V^h. \tag{3.1.11}$$

The first inequality on the left is the inverse inequality due to the partition being quasi-uniform (see Ciarlet [13] page 142). The second inequality follows from the first and (3.1.6). The third follows from noting $|\mathcal{G}^h v^h|_1 \leq |\mathcal{G}v^h|_1$. The final inequality follows from noting (3.1.8) with $m = 0$ and the second inequality above.

We define P^h to be discrete L^2 projection onto S^h: Given $\eta \in L^2(\Omega)$, $P^h\eta$ is a unique solution of

$$(P^h\eta, \xi)^h = (\eta, \xi) \quad \forall \xi \in S^h.$$

This projection satisfies the following bound (see Blowey and Elliott [10] page 155)

$$\|(I - P^h)\eta\|_{-1} \leq Ch|\eta|_0 \quad \forall \eta \in L^2(\Omega). \tag{3.1.12}$$

Given $u_1^0, u_2^0 \in H^1(\Omega)$, let

$$m_1 := \int_\Omega u_1^0 dx, \qquad m_2 := \int_\Omega u_2^0 dx,$$

and take $U_1^0 = P^h u_1^0$ and $U_2^0 = P^h u_2^0$. Then automatically we have

$$(U_1^0, 1)^h = m_1, \qquad (U_2^0, 1)^h = m_2. \tag{3.1.13}$$

Given $\eta \in H^1(\Omega)$, $P_1^h\eta$ is the projection onto S^h such that

$$(P_1^h\eta, 1) = (\eta, 1),$$

and

$$(\nabla P_1^h\eta, \nabla\xi) = (\nabla\eta, \nabla\xi) \quad \forall \xi \in S^h.$$

Notice that due to the nature of projections, it follows that

$$P_1^h\eta \to \eta \quad \text{in } H^1(\Omega) \text{ strongly} \quad \text{and} \quad |P_1^h\eta|_1 \leq |\eta|_1. \tag{3.1.14}$$

In the next section we prove the existence and uniqueness, however we complete this section with some preliminary lemmas, which will prove useful.

Lemma 3.1.1 Let $v \in S^h$, and $r \in \mathbb{R}$, $r \geq 2$ for $d = 1, 2, 3$. Then

$$|(I - \pi^h)v^r|_{0,1} \leq \begin{cases} Ch^2|v|_1^r & \text{for} \quad d = 1, \\ Ch^2\Big(\ln\frac{1}{h}\Big)^{(r-2)/2}|v|_1^r & \text{for} \quad d = 2, \\ Ch^{3-r/2}\|v\|_1^r & \text{for} \quad d = 3. \end{cases} \tag{3.1.15}$$

Proof. Throughout the proof we use the following notation:

$$\mathcal{P}_1(\tau) := \{v : v \text{ is a polynomial of degree } \leq 1 \text{ on } \tau\}.$$

We prove this for each dimension separately.

One Dimensional Case: Let $\eta_i(x)$ and $\eta_{i+1}(x)$ be the nodal basis for $\mathcal{P}_1(I_i)$. Thus for $f \in \mathcal{P}_1(I_i)$, we have

$$f(x) = f(x_i)\eta_i(x) + f(x_{i+1})\eta_{i+1}(x) \quad \forall x \in I_i,$$

and

$$\pi^h f(x) = f(x_i)\eta_i(x) + f(x_{i+1})\eta_{i+1}(x) \quad \forall x \in I_i, \tag{3.1.16}$$

since $\pi^h f(x_i) = f(x_i)$.

The Taylor expansion of $f \in C^2$ about $x \in I_i$ is

$$f(y) = f(x) + (y - x)f'(x) + \frac{(y - x)^2}{2}f''(\xi), \tag{3.1.17}$$

where ξ lies between x and y. Choosing $y = x_i$ and $y = x_{i+1}$ in (3.1.17), we have respectively

$$f(x_i) = f(x) + (x_i - x)f'(x) + \frac{(x_i - x)^2}{2}f''(\xi_i), \tag{3.1.18}$$

$$f(x_{i+1}) = f(x) + (x_{i+1} - x)f'(x) + \frac{(x_{i+1} - x)^2}{2}f''(\xi_{i+1}). \tag{3.1.19}$$

Substituting (3.1.18) and (3.1.19) into (3.1.16), we have for $x \in (x_i, x_{i+1}) = I_i$

$$\pi^h f(x) = f(x) \sum_{j=i}^{i+1} \eta_j(x) + \sum_{j=i}^{i+1} p_j \eta_j(x) + \sum_{j=i}^{i+1} R_j \eta_j(x), \tag{3.1.20a}$$

where

$$p_j = (x_j - x) f'(x), \tag{3.1.20b}$$

$$R_j = R(f; x, x_j) = \frac{(x_j - x)^2}{2} f''(\xi_j). \tag{3.1.20c}$$

Recall that $\pi^h(f) = f$ for $f \in P_1(I_i)$, which follows from the fact that there is a unique function $f \in P_1(I_i)$ assuming given values at the nodes of I_i. Taking $f(x) \equiv 1$ in (3.1.20a) we obtain

$$\sum_{j=i}^{i+1} \eta_j(x) = 1, \tag{3.1.21}$$

since in this case $p_j = R_j = 0$.

Now let $f(x) = \alpha x$, $\alpha \in \mathbb{R}$ in (3.1.20a). Since f is a linear function we have $\pi^h(f) = f$, $p_j = (x_j - x)\alpha$ and $R_j = 0$. Substituting these values into (3.1.20a), we have

$$\sum_{j=i}^{i+1} p_j \eta_j(x) = 0. \tag{3.1.22}$$

Thus noting (3.1.21) and (3.1.22) we can rewrite (3.1.20a) as

$$f - \pi^h f = -\sum_{j=i}^{i+1} R_j \eta_j(x),$$

hence

$$|(I - \pi^h) f(x)|_{0,1,I_i} = \left| \sum_{j=i}^{i+1} R_j \eta_j(x) \right|_{0,1,I_i}.$$

Recalling

$$\eta_j(x) = \begin{cases} \dfrac{x - x_{i+1}}{x_i - x_{i+1}} & \text{for } x \in I_i,\ j = i, \\[2ex] \dfrac{x - x_i}{x_{i+1} - x_i} & \text{for } x \in I_i,\ j = i+1, \end{cases}$$

we have

$$\int_{x_i}^{x_{i+1}} (\eta_j(x))^2 dx = \frac{h_i}{3}.$$

Hence noting the Young inequality (2.1.4) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|(I-\pi^h)f(x)|_{0,1,I_i} &\le \sum_{j=i}^{i+1}\left(\int_{x_i}^{x_{i+1}} |R_j\eta_j(x)|dx\right) \\
&\le \sum_{j=i}^{i+1}\left(\int_{x_i}^{x_{i+1}} (R_j)^2 dx\right)^{\frac{1}{2}}\left(\int_{x_i}^{x_{i+1}} (\eta_j(x))^2 dx\right)^{\frac{1}{2}} \\
&\le Ch_i^{\frac{1}{2}}\sum_{j=i}^{i+1}|R_j|_{0,2,I_i}. \qquad (3.1.23)
\end{aligned}$$

Using the Cauchy-Schwarz inequality we can bound $|R_j|_{0,2,I_i}$ as follows

$$\begin{aligned}
|R_j|_{0,2,I_i} &\le \left|\frac{(x_j-x)^2}{2}f''(\xi_j)\right|_{0,2,I_i} \\
&= \frac{1}{2}\left|(x_j-x)^2 f''(\xi_j)\right|_{0,2,I_i} \\
&\le \frac{1}{2}\left(\int_{I_i}(x_j-x)^8 dx\right)^{\frac{1}{4}}\left(\int_{I_i}(f''(\xi_j))^4 dx\right)^{\frac{1}{4}} \\
&\le \frac{1}{2}h_i^{\frac{9}{4}}\ |f''(\xi_j)|_{0,4,I_i}. \qquad (3.1.24)
\end{aligned}$$

Now consider $f(x) = v^r(x)$, where $v(x) \in S^h$ and $r \in \mathbb{R}$, $r \ge 2$, then for $x \in I_i$, $f'(x) = rv^{r-1}(x)v'(x)$, and $f''(x) = r(r-1)v^{r-2}(x)(v'(x))^2$, since $v''(x) = 0$. Substituting (3.1.24) into (3.1.23) we have

$$\begin{aligned}
|(I-\pi^h)v^r|_{0,1,I_i} &\le Ch_i^{\frac{11}{4}}\ \sum_{j=i}^{i+1}|v^{r-2}(\xi_j)(v'(\xi_j))^2|_{0,4,I_i} \\
&\le Ch_i^{\frac{11}{4}}\ \|v\|_{0,\infty,I_i}^{r-2}\sum_{j=i}^{i+1}|v'(\xi_j)|_{0,8,I_i}^2 \\
&= Ch_i^{\frac{11}{4}}\ \|v\|_{0,\infty,I_i}^{r-2}|v|_{1,8,I_i}^2 \\
&\le Ch_i^2\ \|v\|_{0,\infty,I_i}^{r-2}|v|_{1,2,I_i}^2, \qquad (3.1.25)
\end{aligned}$$

where we have noted the inverse inequality (3.1.9), for $d = 1$, $p_1 = 2$, $p_2 = 8$, and $m = 1$ to obtain the last inequality.

Noting the Sobolev embedding result (3.1.10) for $d = 1$, and summing (3.1.25) over I_i we have

$$\begin{aligned} |(I-\pi^h)v^r|_{0,1,\Omega} &= \sum_{I_i} |(I-\pi^h)v^r|_{0,1,I_i} \\ &\le Ch^2 \sum_{I_i} \|v\|_{0,\infty,I_i}^{r-2} \, |v|_{1,2,I_i}^2 \\ &\le Ch^2 \, \|v\|_{0,\infty,\Omega}^{r-2} \sum_{I_i} |v|_{1,2,I_i}^2 \\ &\le Ch^2 |v|_{1,2,\Omega}^r. \end{aligned}$$

Two Dimensional Case: Now consider a triangle $\tau \in \mathcal{T}^h$ having a local node points $a^k = (a_1^k, a_2^k)$, $k = 1, 2, 3$. Let $\eta_k(x)$ be local basis for $\mathcal{P}_1(\tau)$. Thus, in a similar fashion to the one dimensional case, for $f \in \mathcal{P}_1(\tau)$, we have

$$f(x) = \sum_{k=1}^{3} f(a^k)\eta_k(x) \quad \forall x \in \tau,$$

and

$$\pi^h f(x) = \sum_{k=1}^{3} f(a^k)\eta_k(x) \quad \forall x \in \tau, \tag{3.1.26}$$

since $\pi^h f(a^k) = f(a^k)$. Using a Taylor expansion about $x = (x_1, x_2) \in \tau$, we have

$$f(y) = f(x) + p(f; x, y) + R(f, x, y), \tag{3.1.27a}$$

where

$$p(f; x, y) = \sum_{j=1}^{2} \frac{\partial f(x)}{\partial x_j}(y_j - x_j), \tag{3.1.27b}$$

$$R(f; x, y) = \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f(\xi)}{\partial x_i \partial x_j}(y_i - x_i)(y_j - x_j), \tag{3.1.27c}$$

and ξ is the point on the line segment between x and y. Choosing $y = a^k$ for $k = 1, 2, 3$ on (3.1.27a), we have

$$f(a^k) = f(x) + p(f; x, a^k) + R(f, x, a^k). \tag{3.1.28}$$

Substituting (3.1.28) into (3.1.26) we have

$$\pi^h f(x) = f(x) \sum_{k=1}^{3} \eta_k(x) + \sum_{k=1}^{3} p(f; x, a^k)\eta_k(x) + \sum_{k=1}^{3} R(f; x, a^k)\eta_k(x). \tag{3.1.29}$$

Note that $\pi^h(f) = f$ for $f \in P_1(\tau)$ assuming given value at the nodes of τ. Adopting the approach of Johnson in [24], taking $f(x) \equiv 1$ in (3.1.29), we obtain

$$\sum_{k=1}^{3} \eta_k(x) = 1, \tag{3.1.30}$$

since in this case $\pi^h(f) = f$ and $p(f; x, a^k) = R(f, x, a^k) = 0$.

Now let $f(x) = d_1x_1 + d_2x_2$, $d_1, d_2 \in \mathbb{R}$ in (3.1.29). Since f is a linear function we have $\pi^h(f) = f$,

$$p(f; x, a^k) = d_1(a_1^k - x_1) + d_2(a_2^k - x_2),$$

and $R(f, x, a^k) = 0$. Substituting these values into (3.1.29) we obtain

$$\sum_{k=1}^{3} [d_1(a_1^k - x_1) + d_2(a_2^k - x_2)]\eta_k(x) = 0.$$

Choosing $d_i = \partial f(x)/\partial x_i$, for $i = 1, 2$, we have

$$\sum_{k=1}^{3} [\frac{\partial f(x)}{\partial x_1}(a_1^k - x_1) + \frac{\partial f(x)}{\partial x_2}(a_2^k - x_2)]\eta_k(x) = 0. \tag{3.1.31}$$

Thus by (3.1.30) and (3.1.31) we can express (3.1.29) as

$$\pi^h f(x) = f(x) + \sum_{k=1}^{3} R(f; x, a^k)\eta_k(x). \tag{3.1.32}$$

Following the approach of the one dimensional case closely, we are able to show that

$$\begin{aligned} |(I-\pi^h)f(x)|_{0,1,\tau} &\le \sum_{k=1}^{3} \left|R(f;x,a^k)\eta_k(x)\right|_{0,1,\tau} \\ &\le \sum_{k=1}^{3} \left(\int_\tau \left(R(f;x,a^k)\right)^2 dx \right)^{\frac{1}{2}} \left(\int_\tau \eta_k^2(x) dx \right)^{\frac{1}{2}} \\ &= \sum_{k=1}^{3} \left|R(f;x,a^k)\right|_{0,2,\tau} \left|\eta_k(x)\right|_{0,2,\tau} \\ &\le C h_\tau \sum_{k=1}^{3} \left|R(f;x,a^k)\right|_{0,2,\tau}, \end{aligned} \tag{3.1.33}$$

where we have noted the following formula (see [23] page 145)

$$\int_\tau \eta_1^\alpha(x)\eta_2^\beta(x)\eta_3^\gamma(x)\, d\tau = \frac{2\ \alpha!\ \beta!\ \gamma!}{(\alpha+\beta+\gamma+2)!}\ area(\tau),$$

to compute $|\eta_k(x)|_{0,2,\tau}$.

Since

$$|(a_i^k - x_i)(a_j^k - x_j)|_{0,4,\tau} = \left(\int_\tau (a_i^k - x_i)^4 (a_j^k - x_j)^4 dx \right)^{\frac{1}{4}} \le C h_\tau^{\frac{5}{2}},$$

we have

$$\begin{aligned} \left|R(f;x,a^k)\right|_{0,2,\tau} &= \frac{1}{2} \left| \sum_{i,j=1}^{2} \frac{\partial^2 f(\xi)}{\partial x_i \partial x_j} (a_i^k - x_i)(a_j^k - x_j) \right|_{0,2,\tau} \\ &\le \frac{1}{2} \sum_{i,j=1}^{2} \left| \frac{\partial^2 f(\xi)}{\partial x_i \partial x_j} (a_i^k - x_i)(a_j^k - x_j) \right|_{0,2,\tau} \\ &= \frac{1}{2} \sum_{i,j=1}^{2} \left(\int_\tau \left(\frac{\partial^2 f(\xi)}{\partial x_i \partial x_j} \right)^2 (a_i^k - x_i)^2 (a_j^k - x_j)^2 dx \right)^{\frac{1}{2}} \\ &\le \frac{1}{2} \sum_{i,j=1}^{2} \left(\int_\tau \left(\frac{\partial^2 f(\xi)}{\partial x_i \partial x_j} \right)^4 dx \right)^{\frac{1}{4}} \left(\int_\tau (a_i^k - x_i)^4 (a_j^k - x_j)^4 dx \right)^{\frac{1}{4}} \\ &\le C h_\tau^{\frac{5}{2}} \sum_{i,j=1}^{2} \left| \frac{\partial^2 f(\xi)}{\partial x_i \partial x_j} \right|_{0,4,\tau}. \end{aligned} \tag{3.1.34}$$

Now consider $f(x) = v^r(x)$, where $v(x) \in S^h$, $r \in \mathbb{R}$, $r \geq 2$ and $x = (x_1, x_2)$, then for $i, j = 1, 2$,

$$\frac{\partial f(x)}{\partial x_i} = r v^{r-1}(x) \frac{\partial}{\partial x_i} v(x),$$
$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = r(r-1) v^{r-2}(x) \Big(\frac{\partial}{\partial x_i} v(x) \Big) \Big(\frac{\partial}{\partial x_j} v(x) \Big).$$

Recall that $\partial v / \partial x_j$ is a constant. Hence substituting (3.1.34) into (3.1.33), we have

$$\begin{aligned}
|(I - \pi^h) v^r|_{0,1,\tau} &\leq C h_\tau^{\frac{7}{2}} \sum_{i,j=1}^{2} \left| v^{r-2}(\xi) \frac{\partial v(\xi)}{\partial x_i} \frac{\partial v(\xi)}{\partial x_j} \right|_{0,4,\tau} \\
&\leq C h_\tau^{\frac{7}{2}} \|v\|_{0,\infty,\tau}^{r-2} \sum_{i,j=1}^{2} \left| \frac{\partial v(\xi)}{\partial x_i} \frac{\partial v(\xi)}{\partial x_j} \right|_{0,4,\tau} \\
&= C h_\tau^{\frac{7}{2}} \|v\|_{0,\infty,\tau}^{r-2} \sum_{i,j=1}^{2} \left(\int_\tau \Big(\frac{\partial v(\xi)}{\partial x_i} \Big)^4 \Big(\frac{\partial v(\xi)}{\partial x_j} \Big)^4 dx \right)^{\frac{1}{4}} \\
&\leq C h_\tau^{\frac{7}{2}} \|v\|_{0,\infty,\tau}^{r-2} \sum_{i,j=1}^{2} \left(\int_\tau \Big(\frac{\partial v(\xi)}{\partial x_i} \Big)^8 dx \right)^{\frac{1}{8}} \left(\int_\tau \Big(\frac{\partial v(\xi)}{\partial x_j} \Big)^8 dx \right)^{\frac{1}{8}} \\
&\leq C h_\tau^{\frac{7}{2}} \|v\|_{0,\infty,\tau}^{r-2} \sum_{i,j=1}^{2} \left| \frac{\partial v(\xi)}{\partial x_i} \right|_{0,8,\tau} \left| \frac{\partial v(\xi)}{\partial x_j} \right|_{0,8,\tau} \\
&\leq C h_\tau^2 \|v\|_{0,\infty,\tau}^{r-2} |v|_{1,2,\tau}^2,
\end{aligned} \tag{3.1.35}$$

where we have noted the inverse inequality (3.1.9) with $d = 2$, $p_1 = 2$, $p_2 = 8$, $m = 1$, to obtain the last inequality.

Summing (3.1.35) over $\tau \in \mathcal{T}^h$ and noting the discrete Sobolev embedding result for $d = 2$, we have

$$\begin{aligned}
|(I - \pi^h) v^r|_{0,1,\Omega} &= \sum_{\tau \in \mathcal{T}^h} |(I - \pi^h) v^r|_{0,1,\tau} \leq C h^2 \sum_{\tau \in \mathcal{T}^h} \|v\|_{0,\infty,\tau}^{r-2} \, |v|_{1,2,\tau}^2 \\
&\leq C h^2 \|v\|_{0,\infty,\Omega}^{r-2} \sum_{\tau \in \mathcal{T}^h} |v|_{1,2,\tau}^2 \leq C h^2 \|v\|_{0,\infty,\Omega}^{r-2} \, |v|_{1,2,\Omega}^2 \\
&\leq C h^2 \Big(\ln \frac{1}{h} \Big)^{(r-2)/2} |v|_{1,2,\Omega}^r.
\end{aligned}$$

Three Dimensional Case: Consider a tetrahedron $\tau \in \mathcal{T}^h$ having local node points $a^l = (a_1^l, a_2^l, a_3^l)$, $l = 1, \ldots, 4$. Let $\eta_l(x)$ be local basis for $\mathcal{P}_1(\tau)$. Thus for $f \in \mathcal{P}_1(\tau)$, we have

$$f(x) = \sum_{l=1}^{4} f(a^l)\eta_l(x),$$

and

$$\pi^h f(x) = \sum_{l=1}^{4} f(a^l)\eta_l(x). \tag{3.1.36}$$

Following the approach of the two dimensional case we will obtain

$$\pi^h f(x) = f(x) + \sum_{l=1}^{4} R(f; x, a^l)\eta_l(x),$$

where

$$R(f; x, a^l) = \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f(\xi)}{\partial x_i \partial x_j}(a_i^l - x_i)(a_j^l - x_j).$$

Hence

$$\begin{aligned}
|f(x) - \pi^h f(x)|_{0,1,\tau} &= \left|\sum_{l=1}^{4} R(f; x, a^l)\eta_l(x)\right|_{0,1,\tau} \leq \sum_{l=1}^{4} |R(f; x, a^l)\eta_l(x)|_{0,1,\tau} \\
&\leq \sum_{l=1}^{4} \left(\int_\tau (R(f; x, a^l))^2 dx\right)^{\frac{1}{2}} \left(\int_\tau \eta_l^2(x) dx\right)^{\frac{1}{2}} \\
&= \sum_{l=1}^{4} \left|R(f; x, a^l)\right|_{0,2,\tau} |\eta_l(x)|_{0,2,\tau} \\
&\leq Ch_\tau^{\frac{3}{2}} \sum_{l=1}^{4} \left|R(f; x, a^l)\right|_{0,2,\tau},
\end{aligned} \tag{3.1.37}$$

where we have noted the following formula (see Huebner [23] page 148)

$$\int_\tau \eta_1^\alpha(x)\eta_2^\beta(x)\eta_3^\gamma(x)\eta_4^\delta(x)\, d\tau = \frac{6\ \alpha!\ \beta!\ \gamma!\ \delta!}{(\alpha + \beta + \gamma + 3)!}\ volume(\tau),$$

to compute $|\eta_l(x)|_{0,2,\tau}$.

Since, for $l = 1, \ldots, 4$,

$$\left|(a_i^l - x_i)(a_j^l - x_j)\right|_{0,4,\tau} = \left(\int_\tau (a_i^l - x_i)^4 (a_j^l - x_j)^4 dx\right)^{\frac{1}{4}} \leq Ch_\tau^{\frac{11}{4}},$$

we have

$$
\begin{aligned}
\left|R(f;x,a^l)\right|_{0,2,\tau} &= \frac{1}{2}\left|\sum_{i,j=1}^{3}\frac{\partial^2 f(\xi)}{\partial x_i\partial x_j}(a_i^l-x_i)(a_j^l-x_j)\right|_{0,2,\tau}\\
&\le \frac{1}{2}\sum_{i,j=1}^{3}\left|\frac{\partial^2 f(\xi)}{\partial x_i\partial x_j}(a_i^l-x_i)(a_j^l-x_j)\right|_{0,2,\tau}\\
&= \frac{1}{2}\sum_{i,j=1}^{3}\left(\int_\tau\left(\frac{\partial^2 f(\xi)}{\partial x_i\partial x_j}\right)^2(a_i^l-x_i)^2(a_j^l-x_j)^2dx\right)^{\frac{1}{2}}\\
&\le \frac{1}{2}\sum_{i,j=1}^{3}\left(\int_\tau\left(\frac{\partial^2 f(\xi)}{\partial x_i\partial x_j}\right)^4dx\right)^{\frac{1}{4}}\left(\int_\tau(a_i^l-x_i)^4(a_j^l-x_j)^4dx\right)^{\frac{1}{4}}\\
&\le Ch_\tau^{\frac{11}{4}}\sum_{i,j=1}^{3}\left|\frac{\partial^2 f}{\partial x_i\partial x_j}\right|_{0,4,\tau}. \qquad (3.1.38)
\end{aligned}
$$

Now consider $f(x)=v^r(x)$, where $v(x)\in S^h$, $r\in\mathbb{R}$, $r\ge 2$ and $x=(x_1,x_2,x_3)$, then for $i,j=1,2,3$, we have

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_i} &= rv^r(x)\frac{\partial}{\partial x_i}v(x),\\
\frac{\partial^2 f(x)}{\partial x_i\partial x_j} &= r(r-1)v^{(r-2)}(x)\Big(\frac{\partial}{\partial x_i}v(x)\Big)\Big(\frac{\partial}{\partial x_j}v(x)\Big).
\end{aligned}
$$

Recall that $\partial v/\partial x_j$ is a constant. Hence substituting (3.1.38) into (3.1.37), noting the Cauchy-Schwarz inequality, and the inverse inequality (3.1.9) with $d=3$, $p_1=2$, and $p_2=8$ and $m=1$, respectively, we have

$$
\begin{aligned}
|(I-\pi^h)v^r|_{0,1,\tau} &\le Ch_\tau^{\frac{17}{4}}\sum_{i,j=1}^{3}\left|v^{r-2}(\xi)\frac{\partial v(\xi)}{\partial x_i}\frac{\partial v(\xi)}{\partial x_j}\right|_{0,4,\tau}\\
&\le Ch_\tau^{\frac{17}{4}}\ \|v\|_{0,\infty,\tau}^{r-2}\sum_{i,j=1}^{3}\left(\int_\tau\left(\frac{\partial v(\xi)}{\partial x_i}\right)^4\left(\frac{\partial v(\xi)}{\partial x_j}\right)^4dx\right)^{\frac{1}{4}}\\
&\le Ch_\tau^{\frac{17}{4}}\ \|v\|_{0,\infty,\tau}^{r-2}\sum_{i,j=1}^{3}\left(\int_\tau\left(\frac{\partial v(\xi)}{\partial x_i}\right)^8dx\right)^{\frac{1}{8}}\left(\int_\tau\left(\frac{\partial v(\xi)}{\partial x_j}\right)^8dx\right)^{\frac{1}{8}}\\
&\le Ch_\tau^{\frac{17}{4}}\ \|v\|_{0,\infty,\tau}^{r-2}\sum_{i,j=1}^{3}\left|\frac{\partial v(\xi)}{\partial x_i}\right|_{0,8,\tau}\left|\frac{\partial v(\xi)}{\partial x_j}\right|_{0,8,\tau}\\
&\le Ch_\tau^2\ \|v\|_{0,\infty,\tau}^{r-2}\ |v|_{1,2,\tau}^2. \qquad (3.1.39)
\end{aligned}
$$

Summing (3.1.39) over $\tau \in \mathcal{T}^h$, noting the inverse inequality (3.1.9) with $p_1 = 6$, $p_2 = \infty$, $m = 0$, $d = 3$ and (2.1.8) with $r = 6$, $m = 1$, $d = 3$, $p = 2$ we have

$$\begin{aligned}
|(I - \pi^h)v^r|_{0,1,\Omega} &= \sum_{\tau \in \mathcal{T}^h} |(I - \pi^h)v^r|_{0,1,\tau} \\
&\leq \sum_{\tau \in \mathcal{T}^h} C h_\tau^2 \, \|v\|_{0,\infty,\tau}^{r-2} \, |v|_{1,2,\tau}^2 \\
&\leq C h^2 \, \|v\|_{0,\infty,\Omega}^{r-2} \sum_{\tau \in \mathcal{T}^h} |v|_{1,2,\tau}^2 \\
&\leq C h^2 \, \|v\|_{0,\infty,\Omega}^{r-2} \, \|v\|_{1,2,\Omega}^2 \\
&\leq C h^{3-r/2} \, |v|_{0,6,\Omega}^{r-2} \, \|v\|_{1,2,\Omega}^2 \\
&\leq C h^{3-r/2} \, \|v\|_{1,2,\Omega}^r.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.1.2 Let $v^h \in S^h$, $r \in \mathbb{R}$, $r \geq 2$ for $d = 1, 2$, and $r \in [2\,,\,6]$ for $d = 3$. Then

$$\int_\Omega \pi[(v^h)^r]dx \leq C\|v^h\|_1^r. \tag{3.1.40}$$

Proof. Note that the inequality

$$\begin{aligned}
\int_\Omega \pi^h[(v^h)^r]dx &\leq \int_\Omega |(I - \pi^h)[(v^h)^r]|dx + \int_\Omega (v^h)^r dx, \\
&= |(I - \pi^h)[(v^h)^r]|_{0,1} + |v|_{0,r}^r \quad \forall r,
\end{aligned} \tag{3.1.41}$$

(3.1.15) and (2.1.8) yield

$$\int_\Omega \pi^h[(v^h)^r]dx \leq \begin{cases} Ch^2|v^h|_1^r + C\|v^h\|_1^r & \text{for} \quad d = 1, \\ Ch^2\left(\ln \dfrac{1}{h}\right)^{(r-2)/2}|v^h|_1^r + C\|v^h\|_1^r & \text{for} \quad d = 2, \\ Ch^{3-r/2}\|v\|_1^r + C\|v^h\|_1^r & \text{for} \quad d = 3. \end{cases}$$

Then (3.1.40) follows. □

Lemma 3.1.3 Let $v^h \in S^h$ and $d = 1, 2, 3$. Then

$$|v^h|_{0,4}^2 \leq C\|v^h\|_1^{\frac{7}{4}}\|v^h\|_{-h}^{\frac{1}{4}}. \tag{3.1.42}$$

Proof. It follows from (2.1.8) and (3.1.6). □

Lemma 3.1.4 Let $\eta_i(x) \in S^h, i = 1, \ldots, 4.$ and $d = 1, 2, 3$. Then

$$|(\eta_1\,\eta_2\,\eta_3, \eta_4)^h - (\eta_1\,\eta_2\,\eta_3, \eta_4)| \leq Ch^{2-\frac{d}{3}}\|\eta_1\|_1\;\|\eta_2\|_1\;\|\eta_3\|_1\;\|\eta_4\|_1. \tag{3.1.43}$$

Proof: Noting Theorem 5 in Ciarlet and Raviart [14], we have for $i, j = 1, 2, 3$

$$\begin{aligned} |(\eta_1\,\eta_2\,\eta_3, \eta_4)^h - (\eta_1\,\eta_2\,\eta_3, \eta_4)| &= \left| \int_\Omega (I - \pi^h)\big((\eta_1\,\eta_2\,\eta_3\,\eta_4)(x)\big) \right|, \\ &\leq Ch^2 \sum_{|\alpha|=2} \left| \frac{\partial^2(\eta_1\,\eta_2\,\eta_3\,\eta_4)}{\partial x_i \partial x_j} \right|_{L^1(\Omega)}. \end{aligned} \tag{3.1.44}$$

Now we bound each terms on the right hand side (3.1.44). Without loss of generality, using the generalised Hölder inequality, (2.1.8) and (3.1.9) we obtain, for $i, j = 1, 2, 3$,

$$\begin{aligned} \left| \frac{\partial\chi_1}{\partial x_i}\frac{\partial\chi_2}{\partial x_j}\chi_3\,\chi_4 \right|_{L^1(\Omega)} &= \int_\Omega \frac{\partial\chi_1}{\partial x_i}\frac{\partial\chi_2}{\partial x_j}\chi_3\,\chi_4 dx, \\ &\leq |\chi_1|_{1,3}\;|\chi_2|_{1,3}\;|\chi_3|_{0,6}\;|\chi_4|_{0,6}, \\ &\leq Ch^{-\frac{d}{3}}\|\chi_1\|_1\;\|\chi_2\|_1\;\|\chi_3\|_1\;\|\chi_4\|_1. \end{aligned} \tag{3.1.45}$$

A bound for remaining terms follows by interchanging the χ_k for $k = 1, \ldots, 4$, on (3.1.45). Hence (3.1.43) follows. □

3.2 Existence and Uniqueness

We define the following semidiscrete approximation to the problem (**P**):

($\mathbf{P}^h$) Find $\{u_1^h, u_2^h, w_1^h, w_2^h\} \in S^h \times S^h \times S^h \times S^h$ such that for *a.e.* $t \in (0,T)$

$$\left(\frac{\partial u_1^h}{\partial t}, \eta\right) = -(\nabla w_1^h, \nabla\eta), \tag{3.2.1a}$$

$$(w_1^h, \eta) = (\phi(u_1^h), \eta)^h + \gamma(\nabla u_1^h, \nabla\eta) + 2D(\Psi_1(u_1^h, u_2^h), \eta)^h, \tag{3.2.1b}$$

$$u_1^h(x,0) = P^h u_1^0(x), \tag{3.2.1c}$$

and

$$\left(\frac{\partial u_2^h}{\partial t}, \eta\right) = -(\nabla w_2^h, \nabla\eta), \tag{3.2.1d}$$

$$(w_2^h, \eta) = (\phi(u_2^h), \eta)^h + \gamma(\nabla u_2^h, \nabla\eta) + 2D(\Psi_2(u_1^h, u_2^h), \eta)^h, \tag{3.2.1e}$$

$$u_2^h(x,0) = P^h u_2^0(x), \tag{3.2.1f}$$

where $\phi(\cdot)$, $\Psi_1(\cdot,\cdot)$, $\Psi_2(\cdot,\cdot)$ are given by (1.0.4g), (1.0.4h) and (1.0.4i) respectively.

Using (3.1.4), we can write (3.2.1a) and (3.2.1d), for $i = 1, 2$, as

$$\left(\nabla(\mathcal{G}^h\frac{\partial u_i^h}{\partial t} + w_i^h), \nabla\eta\right) = 0 \quad \forall\eta \in H^1(\Omega). \tag{3.2.2}$$

Taking $\eta = \mathcal{G}^h\frac{\partial u_i^h}{\partial t} + w_i^h$ in (3.2.2), we have for *a.e.* $t \in (0,T)$

$$0 = \left|\mathcal{G}^h\frac{\partial u_i^h}{\partial t} + w_i^h\right|_1^2 = \left|\mathcal{G}^h\frac{\partial u_i^h}{\partial t} + w_i^h - \fint w_i^h\right|_1^2.$$

Thus by the Poincaré inequality (3.1.3) we have

$$0 = \left|\mathcal{G}^h\frac{\partial u_i^h}{\partial t} + w_i^h - \fint w_i^h\right|_1 \geq \widetilde{C}_P^{-1}\left|\mathcal{G}^h\frac{\partial u_i^h}{\partial t} + w_i^h - \fint w_i^h\right|_0.$$

Hence we obtain

$$w_i^h = -\mathcal{G}^h\frac{\partial u_i^h}{\partial t} + \fint w_i^h, \tag{3.2.3}$$

where

$$\fint w_i^h = \frac{1}{|\Omega|}\big((\phi(u_i^h),1)^h + 2D(\Psi_i(u_1^h,u_2^h),1)^h\big). \tag{3.2.4}$$

Noting (3.2.3), (3.2.4) and

$$(\varphi(r),\eta)^h - \frac{1}{|\Omega|}\big((\varphi(r),1)^h,\eta\big) = \big(\varphi(r),(I-\fint\,)\eta\big)^h,$$

we can restate the problem ($\mathbf{P}^h$) as:

Find $\{u_1^h,u_2^h\} \in S^h \times S^h$ such that for $i=1,2$, $u_i^h(0) = P^h u_i^0$ and for *a.e.* $t \in (0,T)$, $(u_i^h(t),1) = (u_i^0,1)$ and

$$(\mathcal{G}^h \frac{\partial u_1^h}{\partial t},\eta) + \gamma(\nabla u_1^h,\nabla\eta) + \big(\phi(u_1^h) + 2D\Psi_1(u_1^h,u_2^h),(I-\fint\,)\eta\big)^h = 0, \tag{3.2.5a}$$

$$(\mathcal{G}^h \frac{\partial u_2^h}{\partial t},\eta) + \gamma(\nabla u_2^h,\nabla\eta) + \big(\phi(u_2^h) + 2D\Psi_2(u_1^h,u_2^h),(I-\fint\,)\eta\big)^h = 0, \tag{3.2.5b}$$

for all $\eta \in S^h$.

Note that taking $\eta = 1$ in (3.2.1a) and (3.2.1d) and integrating over $(0,t)$ we obtain

$$0 = \int_0^t\int_\Omega \frac{\partial u_i^h}{\partial s}dxds = \int_\Omega\int_0^t \frac{\partial u_i^h}{\partial s}dsdx = (u_i^h(t),1) - (u_i^h(0),1).$$

Since u_i^h is piecewise linear, we have

$$(u_i^h(t),1)^h = (u_i^h(t),1) = (u_i^h(0),1) = (u_i^h(0),1)^h = (P^h u_i^0,1)^h = (u_i^0,1), \tag{3.2.6}$$

which implies for any t that

$$|(u_i^h(t),1)^h| \le C. \tag{3.2.7}$$

Theorem 3.2.1 Let the assumptions on u_i^0 of Theorem 2.2.1 and the assumptions **(A)** hold. Then for all $h > 0$ and $d = 1,2,3$, there exists a unique solution $\{u_i^h, w_i^h\}$

to ($\mathbf{P}^{\mathrm{h}}$) such that the following stability bounds hold independently of h:

$$\|u_i^h\|_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{3.2.8a}$$

$$\|u_i^h\|_{H^1(0,T;(H^1(\Omega))')} \le C, \tag{3.2.8b}$$

$$\|w_i^h\|_{L^2(0,T;H^1(\Omega))} \le C(1+T^{\frac{1}{2}}). \tag{3.2.8c}$$

Proof. We write (3.2.1a–c) and (3.2.1d–f) using the representation

$$u_i^h(t,x) = \sum_{j=1}^{J} \widehat{c}_{ij}(t)\eta_j(x), \tag{3.2.9a}$$

$$w_i^h(t,x) = \sum_{j=1}^{J} \widehat{d}_{ij}(t)\eta_j(x), \tag{3.2.9b}$$

with $\widehat{c}_{ij}(t),\ \widehat{d}_{ij}(t) \in \mathbb{R}$. Using (3.2.9a–b) and taking $\eta = \eta_j,\ j = 1,\ldots,J,$ in (3.2.1a–c) and (3.2.1d–f), we obtain

$$\sum_{i=1}^{J} \frac{d\widehat{c}_{1i}}{dt}(\eta_i,\eta_j) = -\sum_{i=1}^{J} \widehat{d}_{1i}(\nabla\eta_i,\nabla\eta_j), \tag{3.2.10a}$$

$$\sum_{i=1}^{J} \widehat{d}_{1i}(\eta_i,\eta_j) = (\phi(u_1^h),\eta_j)^h + \gamma\sum_{i=1}^{J} \widehat{c}_{1i}(\nabla\eta_i,\nabla\eta_j) + 2D(\Psi_1(u_1^h,u_2^h),\eta_j)^h, \tag{3.2.10b}$$

$$\sum_{i=1}^{J} \widehat{c}_{1i}(0)(\eta_i,\eta_j) = (P^h u_1^0,\eta_j), \tag{3.2.10c}$$

$$\sum_{i=1}^{J} \frac{d\widehat{c}_{2i}}{dt}(\eta_i,\eta_j) = -\sum_{i=1}^{J} \widehat{d}_{2i}(\nabla\eta_i,\nabla\eta_j), \tag{3.2.10d}$$

$$\sum_{i=1}^{J} \widehat{d}_{2i}(\eta_i,\eta_j) = (\phi(u_2^h),\eta_j)^h + \gamma\sum_{i=1}^{J} \widehat{c}_{2i}(\nabla\eta_i,\nabla\eta_j) + 2D(\Psi_2(u_1^h,u_2^h),\eta_j)^h, \tag{3.2.10e}$$

$$\sum_{i=1}^{J} \widehat{c}_{2i}(0)(\eta_i,\eta_j) = (P^h u_2^0,\eta_j), \tag{3.2.10f}$$

or

$$B\frac{d\widehat{\boldsymbol{c}}_1}{dt} = -A\widehat{\boldsymbol{d}}_1,$$
$$B\widehat{\boldsymbol{d}}_1 = f(\widehat{\boldsymbol{c}}_1) + \gamma A\widehat{\boldsymbol{c}}_1 + 2Dg_1(\widehat{\boldsymbol{c}}_1, \widehat{\boldsymbol{c}}_2),$$
$$B\widehat{\boldsymbol{c}}_1(0) = P^h\boldsymbol{u}_1^h,$$
$$B\frac{d\widehat{\boldsymbol{c}}_2}{dt} = -A\widehat{\boldsymbol{d}}_2,$$
$$B\widehat{\boldsymbol{d}}_2 = f(\widehat{\boldsymbol{c}}_2) + \gamma A\widehat{\boldsymbol{c}}_2 + 2Dg_2(\widehat{\boldsymbol{c}}_1, \widehat{\boldsymbol{c}}_2),$$
$$B\widehat{\boldsymbol{c}}_2(0) = P^h\boldsymbol{u}_2^h,$$

where

$$\{B\}_{ij} = (\eta_i, \eta_j),$$
$$\{A\}_{ij} = (\nabla\eta_i, \nabla\eta_j),$$
$$\{f(\widehat{\boldsymbol{c}}_i)\}_j = (\phi(u_i^h), \eta_j)^h,$$
$$\{g_2(\widehat{\boldsymbol{c}}_1, \widehat{\boldsymbol{c}}_2)\}_j = (\Psi_i(u_1^h, u_2^h), \eta_j)^h.$$

Since B is a nonsingular matrix we have

$$\frac{d\widehat{\boldsymbol{c}}_1}{dt} = -B^{-1}AB^{-1}f(\widehat{\boldsymbol{c}}_1) + \gamma B^{-1}AB^{-1}A\widehat{\boldsymbol{c}}_1 + 2DB^{-1}AB^{-1}g_1(\widehat{\boldsymbol{c}}_1, \widehat{\boldsymbol{c}}_2),$$
$$\widehat{\boldsymbol{c}}_1(0) = B^{-1}P^h\boldsymbol{u}_1^h,$$
$$\frac{d\widehat{\boldsymbol{c}}_2}{dt} = -B^{-1}AB^{-1}f(\widehat{\boldsymbol{c}}_2) + \gamma B^{-1}AB^{-1}A\widehat{\boldsymbol{c}}_2 + 2DB^{-1}AB^{-1}g_2(\widehat{\boldsymbol{c}}_1, \widehat{\boldsymbol{c}}_2),$$
$$\widehat{\boldsymbol{c}}_2(0) = B^{-1}P^h\boldsymbol{u}_1^h,$$

Defining $\widehat{\boldsymbol{c}} = [\widehat{\boldsymbol{c}}_1\ ,\ \widehat{\boldsymbol{c}}_2]^T$ and $\boldsymbol{u}^h = [\boldsymbol{u}_1^h\ ,\ \boldsymbol{u}_2^h]^T$ we have

$$\frac{d\widehat{\boldsymbol{c}}}{dt} = \widehat{\mathcal{H}}(\widehat{\boldsymbol{c}}),$$
$$\widehat{\boldsymbol{c}}(0) = \mathcal{B}P^h\boldsymbol{u}^h.$$

It follows from the theory of systems of ordinary differential equations that there exists a unique solution on some time interval for $\boldsymbol{c}$. Hence we have local existence for u_i^h and w_i^h for some $t \in (0, t_m)$. To obtain existence of a global solution, we only need to show that *a priori* estimates of u_i^h, w_i^h independent of h.

Now we derive the bound (3.2.8a). Setting $\eta = \partial u_i^h / \partial t$, for $i = 1, 2$, in (3.2.5a–b) respectively, adding the resulting equations, rearranging the terms and integrating over $(0, t)$, we have for all $t \in (0, T)$

$$
\begin{aligned}
0 = & \int_0^t \left(\mathcal{G}^h \frac{\partial u_1^h}{\partial s}, \frac{\partial u_1^h}{\partial s} \right) ds + \int_0^t \left(\mathcal{G}^h \frac{\partial u_2^h}{\partial s}, \frac{\partial u_2^h}{\partial s} \right) ds \\
& + \gamma \int_0^t \left(\nabla u_1^h, \nabla \frac{\partial u_1^h}{\partial s} \right) ds + \gamma \int_0^t \left(\nabla u_2^h, \nabla \frac{\partial u_2^h}{\partial s} \right) ds \\
& + \int_0^t (\phi(u_1^h), \frac{\partial u_1^h}{\partial s})^h ds + \int_0^t (\phi(u_2^h), \frac{\partial u_2^h}{\partial s})^h ds \\
& + 2D \left(\int_0^t (\Psi_1(u_1^h, u_2^h), \frac{\partial u_1^h}{\partial s})^h ds + \int_0^t (\Psi_2(u_1^h, u_2^h), \frac{\partial u_2^h}{\partial s})^h ds \right),
\end{aligned} \tag{3.2.11}
$$

where we have noted

$$
\fint \frac{\partial u_i^h}{\partial t} = \frac{1}{|\Omega|} (\frac{\partial u_i^h}{\partial t}, 1) = \frac{1}{|\Omega|} \frac{\partial}{\partial t} (u_i^h, 1) = \frac{1}{|\Omega|} \frac{\partial}{\partial t} (u_i^0, 1) = 0. \tag{3.2.12}
$$

Now we examine each term on the right hand side of (3.2.11) in turn. Using (3.1.5) we have for $i = 1, 2$

$$
\int_0^t \left(\mathcal{G}^h \frac{\partial u_i^h}{\partial s}, \frac{\partial u_i^h}{\partial s} \right) ds = \int_0^t \left| \mathcal{G}^h \frac{\partial u_i^h}{\partial s} \right|_1^2 ds = \int_0^t \left\| \frac{\partial u_i^h}{\partial s} \right\|_{-h}^2 ds. \tag{3.2.13}
$$

The third and fourth terms of (3.2.11) can be expressed as

$$
\int_0^t \left(\nabla u_i^h, \nabla \frac{\partial u_i^h}{\partial s} \right) ds = \frac{1}{2} \int_0^t \frac{\partial}{\partial s} (\nabla u_i^h, \nabla u_i^h) ds = \frac{1}{2} |u_i^h(t)|_1^2 - \frac{1}{2} |u_i^h(0)|_1^2. \tag{3.2.14}
$$

while the fifth and sixth terms are

$$
\begin{aligned}
\int_\Omega \pi^h \left[\int_0^t \phi(u_i^h) \frac{\partial}{\partial s} u_i^h ds \right] dx &= \int_\Omega \pi^h \left[\int_0^t \frac{\partial}{\partial s} \psi(u_i^h(s)) \right] dx \\
&= \int_\Omega \pi^h \left[\psi(u_i^h(t)) - \psi(u_i^h(0)) \right] dx \\
&= \left(\psi(u_i^h(t)), 1 \right)^h - \left(\psi(u_i^h(0)), 1 \right)^h.
\end{aligned} \tag{3.2.15}
$$

The last two terms of (3.2.11) can be written as

$$\begin{aligned}
&\int_0^t \left(\Psi_1(u_1^h, u_2^h), \frac{\partial u_1^h}{\partial s}\right)^h ds + \int_0^t \left(\Psi_2(u_1^h, u_2^h), \frac{\partial u_2^h}{\partial s}\right)^h ds \\
&\qquad = \int_0^t \left[\int_\Omega \pi^h \left(\Psi_1(u_1^h, u_2^h)\frac{\partial u_1^h}{\partial s} + \Psi_2(u_1^h, u_2^h)\frac{\partial u_2^h}{\partial s}\right) dx\right] ds \\
&\qquad = \int_\Omega \pi^h \left[\int_0^t \left(\frac{\partial}{\partial u_1^h}\Psi(u_1^h, u_2^h)\frac{\partial u_1^h}{\partial s} + \frac{\partial}{\partial u_2^h}\Psi(u_1^h, u_2^h)\frac{\partial u_2^h}{\partial s}\right) ds\right] dx \\
&\qquad = \int_\Omega \pi^h \left[\int_0^t \frac{\partial}{\partial s}\Psi(u_1^h, u_2^h) ds\right] dx \\
&\qquad = \int_\Omega \pi^h [\Psi(u_1^h(t), u_2^h(t)) - \Psi(u_1^h(0), u_2^h(0))] dx \\
&\qquad = \big(\Psi(u_1^h(t), u_2^h(t)), 1\big)^h - \big(\Psi(u_1^h(0), u_2^h(0)), 1\big)^h. \qquad (3.2.16)
\end{aligned}$$

Substituting (3.2.13–3.2.16) into (3.2.11), noting (3.2.1c) and (3.2.1f), and rearranging the terms we have

$$\begin{aligned}
&\int_0^t \left\|\frac{\partial u_1^h}{\partial s}\right\|_{-h}^2 ds + \int_0^t \left\|\frac{\partial u_2^h}{\partial s}\right\|_{-h}^2 ds + \frac{\gamma}{2}|u_1^h(t)|_1^2 + \frac{\gamma}{2}|u_2^h(t)|_1^2 \\
&\quad + \big(\psi(u_1^h(t)), 1\big)^h + \big(\psi(u_2^h(t)), 1\big)^h + 2D\big(\Psi(u_1^h(t), u_2^h(t)), 1\big)^h \\
&\qquad = \frac{\gamma}{2}|P^h u_1^0|_1^2 + \frac{\gamma}{2}|P^h u_2^0|_1^2 + \big(\psi(u_1^h(0)), 1\big)^h + \big(\psi(u_2^h(0)), 1\big)^h \\
&\qquad\quad + 2D\big(\Psi(u_1^h(0), u_2^h(0)), 1\big)^h. \qquad (3.2.17)
\end{aligned}$$

It follows from (3.1.40) that

$$\begin{aligned}
(\psi(u_i^h(0)), 1)^h &= \frac{1}{4}\big(((u_i^h(0))^2 - 1)^2, 1\big)^h \\
&\le \frac{1}{4}\big((u_i^h(0))^4 + 1, 1\big)^h \\
&= \frac{1}{4}\int_\Omega \pi^h[(u_i^h(0))^4] dx + \frac{1}{4}\int_\Omega dx \\
&\le C\|u_i^h(0)\|_1^4 + \frac{1}{4}|\Omega| \\
&= C\|P^h u_i^0\|_1^4 + \frac{1}{4}|\Omega|. \qquad (3.2.18)
\end{aligned}$$

The Young inequality (2.1.4), and (3.1.40) yield

$$
\begin{aligned}
\left(\Psi(u_1^h(0), u_2^h(0)), 1\right)^h &= \left((u_1^h(0)+1)^2(u_2^h(0)+1)^2, 1\right)^h \\
&\le \frac{1}{2}\left((u_1^h(0)+1)^4, 1\right)^h + \frac{1}{2}\left((u_2^h(0)+1)^4, 1\right)^h \\
&\le \frac{9}{2}\left((u_1^h(0))^4+1, 1\right)^h + \frac{9}{2}\left((u_2^h(0))^4+1, 1\right)^h \\
&\le C\|P^h u_1^0\|_1^4 + C\|P^h u_2^0\|_1^4 + 9|\Omega|. \qquad (3.2.19)
\end{aligned}
$$

Substituting (3.2.18)–(3.2.19) into (3.2.17), noting (3.2.1c) and (3.2.1f), and simplifying we have

$$
\begin{aligned}
&\int_0^t \left\|\frac{\partial u_1^h}{\partial s}\right\|_{-h}^2 ds + \int_0^t \left\|\frac{\partial u_2^h}{\partial s}\right\|_{-h}^2 ds + \frac{\gamma}{2}|u_1^h(t)|_1^2 + \frac{\gamma}{2}|u_2^h(t)|_1^2 \\
&+ \left(\psi(u_1^h(t)), 1\right)^h + \left(\psi(u_2^h(t)), 1\right)^h + 2D\left(\Psi(u_1^h(t), u_2^h(t)), 1\right)^h \\
&= C\|P^h u_1^0\|_1^4 + C\|P^h u_2^0\|_1^4 + (18D + \frac{1}{2})|\Omega| \le C, \qquad (3.2.20)
\end{aligned}
$$

where C is independent of T. Using the Poincaré inequality (3.1.3), (3.2.20) and (3.2.7) we obtain

$$
|u_i^h(t)|_0 \le C_P(|u_i^h(t)|_1 + |(u_i^h(t), 1)^h|) \le C. \qquad (3.2.21)
$$

It follows that $u_i^h(t) \in H^1(\Omega)$. Hence

$$
\|u_i^h(t)\|_{L^\infty(0,T;H^1(\Omega))} \le C. \qquad (3.2.22)
$$

Next we show the bound (3.2.8b). Noting (3.2.3) and (3.1.5) we obtain

$$
|w_i^h|_1^2 = \left| -\mathcal{G}^h \frac{\partial u_i^h}{\partial t} + \fint w_i^h \right|_1^2 = \left|\mathcal{G}^h \frac{\partial u_i^h}{\partial t}\right|_1^2 = \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-h}^2. \qquad (3.2.23)
$$

Hence setting $t = T$ in (3.2.20) we obtain

$$
\begin{aligned}
&\int_0^T |w_1^h(t)|_1^2 dt + \int_0^T |w_2^h(t)|_1^2 dt + \frac{\gamma}{2}|u_1^h(T)|_1^2 + \frac{\gamma}{2}|u_2^h(T)|_1^2 + (\psi(u_1^h(T)), 1)^h \\
&+ (\psi(u_2^h(T)), 1)^h + (\Psi(u_1^h(T), u_2^h(T)), 1)^h \le C, \qquad (3.2.24)
\end{aligned}
$$

and in particular

$$\int_0^T \left\| \frac{\partial u_1^h}{\partial t} \right\|_{-h}^2 ds + \int_0^T \left\| \frac{\partial u_2^h}{\partial t} \right\|_{-h}^2 ds \le C. \tag{3.2.25}$$

This implies using (3.1.11) that

$$\left\| \frac{\partial u_i^h}{\partial t} \right\|_{L^2(0,T;(H^1(\Omega))')} \le C. \tag{3.2.26}$$

Recall that the mass is conserved. This allows us to show u_i^h is bounded in $L^2(0,T;(H^1(\Omega))')$ by showing $u_i^h - \fint u_i^h \in L^2(0,T;(H^1(\Omega))')$. Consider

$$\begin{aligned}\left\| u_i^h - \fint u_i^h \right\|_{-h}^2 &= \| u_i^h(t) - u_i^h(0) + u_i^h(0) - \frac{1}{|\Omega|}(u_i^h(t),1) \|_{-h}^2 \\ &= \left\| \int_0^t \frac{\partial u_i^h(t)}{\partial s} ds + u_i^h(0) - \frac{1}{|\Omega|}(u_i^h(t),1) \right\|_{-h}^2 . \end{aligned} \tag{3.2.27}$$

Noting the Young inequality (2.1.4), (3.1.7), (3.2.6) and taking $t = T$ in (3.2.27) we obtain

$$\begin{aligned}\left\| u_i^h - \fint u_i^h \right\|_{-h}^2 &\le 2 \int_0^T \left\| \frac{\partial u_i^h(t)}{\partial s} \right\|_{-h}^2 ds + 2 \left\| u_i^h(0) - \frac{1}{|\Omega|}(u_i^h(t),1) \right\|_{-h}^2 \\ &\le 2 \int_0^T \left\| \frac{\partial u_i^h(t)}{\partial s} \right\|_{-h}^2 ds + C|u_i^h(0)|_0^2 + C|(u_i^h(t),1)|_0^2 \\ &\le 2 \int_0^T \left\| \frac{\partial u_i^h(t)}{\partial s} \right\|_{-h}^2 ds + C \le C, \end{aligned} \tag{3.2.28}$$

where we have noted (3.2.25) and the condition on u_i^0.

Integrating (3.2.28) over $(0,T)$ we obtain

$$\left\| u_i^h - \fint u_i^h \right\|_{L^2(0,T;(H^1(\Omega))')} < C(T) < C. \tag{3.2.29}$$

Hence (3.2.25) and (3.2.29) imply that

$$\|u_i^h\|_{H^1(0,T;(H^1(\Omega))')} < C. \tag{3.2.30}$$

Now we show that $w_i^h \in H^1(\Omega)$. Setting $\xi = w_i^h$ in the Poincaré inequality (2.1.2), and noting the Young inequality (2.1.4) we have

$$|w_i^h(t)|_0^2 \leq C(|w_i^h(t)|_1^2 + |(w_i^h(t), 1)|^2). \tag{3.2.31}$$

The definition of the norm in H^1 and (3.2.31) yield

$$\|w_i^h(t)\|_1^2 \leq C(|w_i^h(t)|_1^2 + |(w_i^h(t), 1)|^2). \tag{3.2.32}$$

Taking $\eta = 1$ in (3.2.1b) and (3.2.1e) we have

$$|(w_i^h(t), 1)| \leq |(\phi(u_i^h(t)), 1)^h| + |(\Psi_i(u_1^h(t), u_2^h(t)), 1)^h|. \tag{3.2.33}$$

Now we bound the right hand side terms of (3.2.33) in turn. Noting the Young inequality (2.1.4), (3.1.2a), (3.1.40) and (3.2.22) we obtain

$$\begin{aligned}
|(\phi(u_i^h(t)), 1)^h| &= \left| \int_\Omega \pi^h[(u_i^h(t))^3 - u_i^h(t)]dx \right| \\
&\leq \frac{1}{2} \int_\Omega \pi^h[((u_i^h(t))^2 - 1)^2]dx + \frac{1}{2} \int_\Omega \pi^h[(u_i^h(t))^2]dx \\
&\leq \int_\Omega \pi^h[(u_i^h(t))^4 + 1]dx + \frac{1}{2}|u_i^h(t)|_h^2 \\
&\leq \int_\Omega \pi^h[(u_i^h(t))^4]dx + |\Omega| + C|u_i^h(t)|_0^2 \\
&\leq C(h)\|u_i^h(t)\|_1^4 + C\|u_i^h(t)\|_1^2 + |\Omega| \leq C.
\end{aligned} \tag{3.2.34}$$

Using the Young inequality (2.1.4), (3.1.2a), (3.1.40) and (3.2.22) we have

$$\begin{aligned}
|(\Psi_1(u_1^h(t), u_2^h(t)), 1)^h| &= |((u_1^h(t) + 1)(u_2^h(t) + 1)^2, 1)^h| \\
&= \left| \int_\Omega \pi^h[(u_1^h(t) + 1)(u_2^h(t) + 1)^2]dx \right| \\
&\leq \frac{1}{2} \int_\Omega \pi^h[(u_1^h(t) + 1)^2 + (u_2^h(t) + 1)^4]dx \\
&\leq \int_\Omega \pi^h[(u_1^h(t))^2 + 4(u_2^h(t))^4 + 5]dx \\
&\leq C|u_1^h(t)|_0^2 + C(h)\|u_2^h(t)\|_1^4 + 5|\Omega| \leq C.
\end{aligned} \tag{3.2.35a}$$

Similarly we have

$$|(\Psi_2(u_1^h(t), u_2^h(t)), 1)^h| \leq C|u_2^h(t)|_0^2 + C(h)\|u_1^h(t)\|_1^4 + 5|\Omega| \leq C. \tag{3.2.35b}$$

Substituting (3.2.34) and (3.2.35a-b) into (3.2.33), then inserting the resulting equations into (3.2.32) we end up with

$$\|w_i^h(t)\|_1^2 \leq C(|w_i^h(t)|_1^2 + C). \tag{3.2.36}$$

Integrating (3.2.36) over $(0, T)$ and noting (3.2.22), we conclude that

$$\|w_i^h(t)\|_{L^2(0,T;H^1(\Omega))} \leq C(1 + T^{\frac{1}{2}}). \tag{3.2.37}$$

Now we are in the position to show the uniqueness. Let $\{u_1^{h,1}, w_1^{h,1}, u_2^{h,1}, w_2^{h,1}\}$ and $\{u_1^{h,2}, w_1^{h,2}, u_2^{h,2}, w_2^{h,2}\}$ be two solutions of $(\mathbf{P}^h)$. Define

$$z_1^h = u_1^{h,1} - u_1^{h,2}, \qquad z_2^h = u_2^{h,1} - u_2^{h,2}. \tag{3.2.38}$$

Substitute these solutions into (3.2.5a–b) we have for $i = 1, 2$,

$$(\mathcal{G}^h \frac{\partial u_1^{h,i}}{\partial t}, \eta) + \gamma(\nabla u_1^{h,i}, \nabla \eta) + (\phi(u_1^{h,i}), \eta)^h + 2D(\Psi_i(u_1^{h,i}, u_2^{h,i}), \eta)^h = 0, \tag{3.2.39}$$

$$(\mathcal{G}^h \frac{\partial u_2^{h,i}}{\partial t}, \eta) + \gamma(\nabla u_2^{h,i}, \nabla \eta) + (\phi(u_2^{h,i}), \eta)^h + 2D(\Psi_i(u_1^{h,i}, u_2^{h,i}), \eta)^h = 0. \tag{3.2.40}$$

Subtracting (3.2.40) from (3.2.39) and summing the resulting equation for $i = 1, 2$, with $\eta = u_1^{h,1} - u_1^{h,2}$ and $\eta = u_2^{h,1} - u_2^{h,2}$ respectively, we obtain

$$\begin{aligned}
(\mathcal{G}^h \frac{\partial z_1^h}{\partial t}, z_1^h) &+ (\mathcal{G}^h \frac{\partial z_2^h}{\partial t}, z_2^h) + \gamma(\nabla z_1^h, \nabla z_1^h) + \gamma(\nabla z_2^h, \nabla z_2^h) \\
&= \big(\phi(u_1^{h,2}) - \phi(u_1^{h,1}), u_1^{h,1} - u_1^{h,2}\big)^h + \big(\phi(u_2^{h,2}) - \phi(u_2^{h,1}), u_2^{h,1} - u_2^{h,2}\big)^h \\
&\quad + 2D\big(\Psi_1(u_1^{h,2}, u_2^{h,2}) - \Psi_1(u_1^{h,1}, u_2^{h,1}), u_1^{h,1} - u_1^{h,2}\big)^h \\
&\quad + 2D\big(\Psi_2(u_1^{h,2}, u_2^{h,2}) - \Psi_2(u_1^{h,1}, u_2^{h,1}), u_2^{h,1} - u_2^{h,2}\big)^h,
\end{aligned} \tag{3.2.41}$$

where we have noted (3.2.38) for the terms on the left hand side.

Noting (2.2.55), (3.1.2a) and (3.1.6) we have

$$
\begin{aligned}
\big(\phi(u_1^{h,2}) - \phi(u_1^{h,1}), u_1^{h,1} - u_1^{h,2}\big)^h &+ \big(\phi(u_2^{h,2}) - \phi(u_2^{h,1}), u_2^{h,1} - u_2^{h,2}\big)^h \\
&= \big((u_1^{h,2})^3 - (u_1^{h,1})^3 + u_1^{h,1} - u_1^{h,2}, u_1^{h,1} - u_1^{h,2}\big)^h \\
&\quad + \big((u_2^{h,2})^3 - (u_2^{h,1})^3 + u_2^{h,1} - u_2^{h,2}, u_2^{h,1} - u_2^{h,2}\big)^h \\
&\leq (u_1^{h,1} - u_1^{h,2}, u_1^{h,1} - u_1^{h,2})^h + (u_2^{h,1} - u_2^{h,2}, u_2^{h,1} - u_2^{h,2})^h \\
&= \|z_1^h\|_h^2 + \|z_2^h\|_h^2 \\
&\leq C\|z_1^h\|_0^2 + C\|z_2^h\|_0^2 \\
&= C(\nabla z_1^h, \nabla \mathcal{G}^h z_1^h) + C(\nabla z_2^h, \nabla \mathcal{G}^h z_2^h) \\
&\leq C|z_1^h|_1 \, \|z_1^h\|_{-h} + C|z_2^h|_1 \, \|z_2^h\|_{-h} \\
&\leq \frac{\alpha}{2}(|z_1^h|_1^2 + |z_2^h|_1^2) + C(\|z_1^h\|_{-h}^2 + \|z_2^h\|_{-h}^2).
\end{aligned} \tag{3.2.42}
$$

Now the Taylor expansion of Ψ about $(u_1^{h,2}, u_2^{h,2})$ and $(u_1^{h,1}, u_2^{h,1})$ are respectively given by

$$
\begin{aligned}
\Psi(u_1^{h,1}, u_2^{h,1}) = \Psi(u_1^{h,2}, u_2^{h,2}) &+ \frac{\partial \Psi(u_1^{h,2}, u_2^{h,2})}{\partial u_1}(u_1^{h,1} - u_1^{h,2}) + \frac{\partial \Psi(u_1^{h,2}, u_2^{h,2})}{\partial u_2}(u_2^{h,1} - u_2^{h,2}) \\
&+ \frac{1}{2}\frac{\partial^2 \Psi(\zeta_1, \zeta_2)}{\partial u_1^2}(u_1^{h,1} - u_1^{h,2})^2 + \frac{\partial^2 \Psi(\zeta_1, \zeta_2)}{\partial u_1 \partial u_2}(u_1^{h,1} - u_1^{h,2})(u_2^{h,1} - u_2^{h,2}) \\
&+ \frac{1}{2}\frac{\partial^2 \Psi(\zeta_1, \zeta_2)}{\partial u_2^2}(u_2^{h,1} - u_2^{h,2})^2,
\end{aligned} \tag{3.2.43a}
$$

and

$$
\begin{aligned}
\Psi(u_1^{h,2}, u_2^{h,2}) = \Psi(u_1^{h,1}, u_2^{h,1}) &+ \frac{\partial \Psi(u_1^{h,1}, u_2^{h,1})}{\partial u_1}(u_1^{h,2} - u_1^{h,1}) + \frac{\partial \Psi(u_1^{h,1}, u_2^{h,1})}{\partial u_2}(u_2^{h,2} - u_2^{h,1}) \\
&+ \frac{1}{2}\frac{\partial^2 \Psi(\xi_1, \xi_2)}{\partial u_1^2}(u_1^{h,2} - u_1^{h,1})^2 + \frac{\partial^2 \Psi(\xi_1, \xi_2)}{\partial u_1 \partial u_2}(u_1^{h,2} - u_1^{h,1})(u_2^{h,2} - u_2^{h,1}) \\
&+ \frac{1}{2}\frac{\partial^2 \Psi(\xi_1, \xi_2)}{\partial u_2^2}(u_2^{h,2} - u_2^{h,1})^2,
\end{aligned} \tag{3.2.43b}
$$

where ζ_1 and ξ_1 are between $u_1^{h,1}$ and $u_1^{h,2}$, and ζ_2 and ξ_2 are between $u_2^{h,1}$ and $u_2^{h,2}$. Adding (3.2.43a) and (3.2.43b), simplifying, interpolating, integrating over Ω, and noting (1.0.4h) and (1.0.4i) we obtain

$$
\begin{aligned}
0 = & \int_\Omega \pi^h \big[(\Psi_1(u_1^{h,2},u_2^{h,2}) - \Psi_1(u_1^{h,1},u_2^{h,1}))(u_1^{h,1}-u_1^{h,2})\big]dx \\
& + \int_\Omega \pi^h \big[(\Psi_2(u_1^{h,2},u_2^{h,2}) - \Psi_2(u_1^{h,1},u_2^{h,1}))(u_2^{h,1}-u_2^{h,2})\big]dx \\
& + \frac{1}{2}\int_\Omega \pi^h \left[\left(\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_1^{(2)}} + \frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_1^{(2)}}\right)(u_1^{h,1}-u_1^{h,2})^2\right]dx \\
& + \frac{1}{2}\int_\Omega \pi^h \left[\left(\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_2^2} + \frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_2^2}\right)(u_2^{h,1}-u_2^{h,2})^2\right]dx \\
& + \int_\Omega \pi^h \left[\left(\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2} + \frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}\right)(u_1^{h,1}-u_1^{h,2})(u_2^{h,1}-u_2^{h,2})\right]dx,
\end{aligned}
$$

so that using the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
& \big|\big((\Psi_1(u_1^{h,2},u_2^{h,2}) - \Psi_1(u_1^{h,1},u_2^{h,1})),(u_1^{h,1}-u_1^{h,2})\big)^h \\
& \quad +\big((\Psi_2(u_1^{h,2},u_2^{h,2}) - \Psi_2(u_1^{h,1},u_2^{h,1})),(u_2^{h,1}-u_2^{h,2})\big)^h\big| \\
& \quad \le \frac{1}{2}\int_\Omega \pi^h \left[\left(\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_1^{(2)}} + \frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_1^{(2)}}\right)(u_1^{h,1}-u_1^{h,2})^2\right]dx \\
& \quad + \frac{1}{2}\int_\Omega \pi^h \left[\left(\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_2^2} + \frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_2^2}\right)(u_2^{h,1}-u_2^{h,2})^2\right]dx \\
& \quad + \frac{1}{2}\int_\Omega \pi^h \left[\left(\left|\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2}\right| + \left|\frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}\right|\right)(u_1^{h,1}-u_1^{h,2})^2\right]dx \\
& \quad + \frac{1}{2}\int_\Omega \pi^h \left[\left(\left|\frac{\partial^2\Psi(\zeta_1,\zeta_2)}{\partial u_1\partial u_2}\right| + \left|\frac{\partial^2\Psi(\xi_1,\xi_2)}{\partial u_1\partial u_2}\right|\right)(u_2^{h,1}-u_2^{h,2})^2\right]dx. \qquad (3.2.44)
\end{aligned}
$$

Hence using (2.2.59a–f) we can rewrite (3.2.44) as

$$
\begin{aligned}
& \big|\big(\Psi_1(u_1^{h,2},u_2^{h,2}) - \Psi_1(u_1^{h,1},u_2^{h,1}),(u_1^{h,1}-u_1^{h,2})\big)^h \\
& \quad +\big(\Psi_2(u_1^{h,2},u_2^{h,2}) - \Psi_2(u_1^{h,1},u_2^{h,1}),(u_2^{h,1}-u_2^{h,2})\big)^h\big| \\
& \quad \le 2\int_\Omega \pi^h \Big[\big((u_2^{h,1})^2 + (u_2^{h,2})^2 + 1\big)(u_1^{h,1}-u_1^{h,2})^2\Big]dx \\
& \quad + 2\int_\Omega \pi^h \Big[\big((u_1^{h,1})^2 + (u_1^{h,2})^2 + 1\big)(u_2^{h,1}-u_2^{h,2})^2\Big]dx \\
& \quad + 2\int_\Omega \pi^h \Big[\big((u_1^{h,1})^2 + (u_1^{h,2})^2 + (u_2^{h,1})^2 + (u_2^{h,2})^2 + 2\big)(u_1^{h,1}-u_1^{h,2})^2\Big]dx \\
& \quad + 2\int_\Omega \pi^h \Big[\big((u_1^{h,1})^2 + (u_1^{h,2})^2 + (u_2^{h,1})^2 + (u_2^{h,2})^2 + 2\big)(u_2^{h,1}-u_2^{h,2})^2\Big]dx. \qquad (3.2.45)
\end{aligned}
$$

On noting (3.1.2a) and (3.1.6) we have

$$\int_\Omega \pi^h[(u_i^{h,1} - u_i^{h,2})^2]dx = |z_i^h|_h^2 \le C|z_i^h|_0^2 \le \frac{\alpha}{2}|z_i^h|_1^2 + C\|z_i^h\|_{-h}^2. \qquad (3.2.46)$$

Note that using (3.1.41), (3.1.15), the Poincaré inequality (2.1.2), (3.2.6), (3.2.7), (2.1.8), and (3.2.22), we have

$$\left(\int_\Omega \pi^h\left[(u_k^{h,j})^4\right]dx\right)^{\frac{1}{2}} \le \left(|(I-\pi^h)[(u_k^{h,j})^4]|_{0,1} + |u_k^{h,j}|_{0,4}^4\right)^{\frac{1}{2}}$$
$$\le \begin{cases} Ch\ |u_k^{h,j}|_1^2 & \text{for} \quad d=1, \\ Ch\left(\ln\frac{1}{h}\right)^{\frac{1}{2}}|u_k^{h,j}|_1^2 & \text{for} \quad d=2, \\ Ch^{\frac{1}{2}}\ |u_k^{h,j}|_1^2 & \text{for} \quad d=3, \end{cases}$$

and using (3.1.41), (3.1.15), (3.1.42), the Poincaré inequality (2.1.2) and $(z_i^h, 1) = 0$ we obtain

$$\left(\int_\Omega \pi^h\left[(z_i^h)^4\right]dx\right)^{\frac{1}{2}} \le \begin{cases} Ch\ |z_i^h|_1^2 + C\ |z_i^h|_1^{\frac{7}{4}}\ \|z_i^h\|_{-h}^{\frac{1}{4}} & \text{for} \quad d=1, \\ Ch\left(\ln\frac{1}{h}\right)^{\frac{1}{2}}\ |z_i^h|_1^2 + C\ |z_i^h|_1^{\frac{7}{4}}\ \|z_i^h\|_{-h}^{\frac{1}{4}} & \text{for} \quad d=2, \\ Ch^{\frac{1}{2}}\ |z_i^h|_1^2 + C\ |z_i^h|_1^{\frac{7}{4}}\ \|z_i^h\|_{-h}^{\frac{1}{4}} & \text{for} \quad d=3. \end{cases}$$

Hence noting the Cauchy-Schwarz inequality, $h\ln\frac{1}{h} \le C$, (3.2.22), the first inequality in (3.1.11), the Young inequality (2.1.4) with $p = 4/3,\ 8/7,\ q = 4,\ 8$ respectively, (3.1.6) and simplifying we obtain

$$
\begin{aligned}
\int_\Omega \pi^h\big[(u_k^{h,j})^2(z_i^h)^2\big]dx &\le \Big(\int_\Omega \pi^h\big[(u_k^{h,j})^4\big]dx\Big)^{\frac12}\Big(\int_\Omega \pi^h\big[(z_i^h)^4\big]dx\Big)^{\frac12}\\
&\le C|u_k^{h,j}|_1^2 \begin{cases} h^2\ |z_i^h|_1^2 + h\ |z_i^h|_1^{\frac74}\ \|z_i^h\|_{-h}^{\frac14} & \text{for } \ d=1,\\ h^2\ \ln\frac1h\ |z_i^h|_1^2 + h\left(\ln\frac1h\right)^{\frac12}\ |z_i^h|_1^{\frac74}\ \|z_i^h\|_{-h}^{\frac14} & \text{for } \ d=2,\\ h\ |z_i^h|_1^2 + h^{\frac12}\ |z_i^h|_1^{\frac74}\ \|z_i^h\|_{-h}^{\frac14} & \text{for } \ d=3,\end{cases}\\
&\le Ch|z_i^h|_1^2 + C\ |z_i^h|_1^{\frac74}\ \|z_i^h\|_{-h}^{\frac14} \quad \text{for } \ d=1,2,3,\\
&\le C|z_i^h|_0|z_i^h|_1 + \frac{7\epsilon}{8}|z_i^h|_1^2 + C\|z_i^h\|_{-h}^2 \quad \text{for } \ d=1,2,3,\\
&\le C|z_i^h|_1^{\frac32}\ \|z_i^h\|_{-h}^{\frac12} + \frac{7\epsilon}{8}|z_i^h|_1^2 + C\|z_i^h\|_{-h}^2 \quad \text{for } \ d=1,2,3,\\
&\le \frac{13\epsilon}{8}|z_i^h|_1^2 + C\|z_i^h\|_{-h}^2 \quad \text{for } \ d=1,2,3.
\end{aligned}\tag{3.2.47}
$$

Noting

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|z_i^h\|_{-h}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\nabla\mathcal{G}^h z_i^h\right|^2 dx\\
&= 2\int_\Omega \big(\nabla\mathcal{G}^h z_i^h \nabla\mathcal{G}^h \frac{dz_i^h}{dt}\big)dx\\
&= 2\big(\mathcal{G}^h\frac{dz_i^h}{dt}, z_i^h\big),
\end{aligned}\tag{3.2.48}
$$

(3.2.42), (3.2.46), and (3.2.47), we can rewrite (3.2.41) as

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|z_1^h\|_{-h}^2 + \|z_2^h\|_{-h}^2\big) + 2\gamma\big(|z_1^h|_1^2 + |z_2^h|_1^2\big)\\
&\qquad \le \big((1+12D)\alpha + 78D\epsilon\big)\big(|z_1^h|_1^2 + |z_2^h|_1^2\big) + C\big(\|z_1^h\|_{-h}^2 + \|z_2^h\|_{-h}^2\big).
\end{aligned}
$$

Setting $\alpha = \epsilon$ and $\epsilon = 7\gamma/(360D+4), D \ge 0$, and rearranging the terms we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\big(\|z_1^h\|_{-h}^2 + \|z_2^h\|_{-h}^2\big) + \frac{\gamma}{4}\big(|z_1^h|_1^2 + |z_2^h|_1^2\big) \le C\big(\|z_1^h\|_{-h}^2 + \|z_2^h\|_{-h}^2\big). \tag{3.2.49}
$$

Integrating over $t \in (0,T)$ and using a Grönwall inequality we conclude from (3.2.49) that

$$\|z_1^h(t)\|_{-h}^2 + \|z_2^h(t)\|_{-h}^2 + \frac{\gamma}{4}\int_0^t \left(|z_1^h(s)|_1^2 + |z_2^h(s)|_1^2\right) ds \le \|z_1^h(0)\|_{-h}^2 + \|z_2^h(0)\|_{-h}^2 = 0.$$

Noting the Poincaré inequality (2.1.2) and $(z_i^h, 1) = 0$, we obtain the uniqueness of u_i^h. The uniqueness of w_i^h follows from (3.2.3) and (3.2.4). This ends the proof of the existence and uniqueness of the problem $(\mathbf{P}^\mathrm{h})$. □

3.3 Error bound

In this section we shall estimate the difference between the solutions u_i of the coupled pair of Cahn-Hilliard equations (2.2.1a–f) and their semidiscrete approximations u_i^h defined in (3.2.1a–f). For the case $D = 0$, i.e. two decoupled Cahn-Hilliard equations, the error bound has been discussed in Elliott, French and Milner [19]. They showed that the error bound is H^1 optimal.

Theorem 3.3.1 Let the assumptions of Theorem 3.2.1 hold. Then for all $h > 0$ and $d = 1, 2, 3$, we have that

$$\|u - u_i^h\|_{L^2(0,T;H^1(\Omega))} + \|u - u_i^h\|_{L^\infty(0,T;H^1(\Omega)')} \le Ch^2. \tag{3.3.1}$$

Proof. Let be $e_i = u_i - u_i^h$, $e_i^A = u_i - \pi^h u_i$ and $e_i^h = \pi^h u_i - u_i^h$, $i = 1, 2$. Subtracting (3.2.5a) and (3.2.5b) from (2.2.5a) and (2.2.5b) respectively, we have

$$\begin{aligned}(\mathcal{G}\frac{\partial u_i}{\partial t}, \eta) + \gamma(\nabla e_i, \nabla \eta) = {} & (\mathcal{G}^h \frac{\partial u_i^h}{\partial t}, \eta) - \left(\phi(u_i) + 2D\Psi_i(u_1, u_2), (I - ⨍\,)\eta\right) \\ & + \left(\phi(u_i^h) + 2D\Psi_i(u_1^h, u_2^h), (I - ⨍\,)\eta\right)^h, \end{aligned} \tag{3.3.2}$$

for all $\eta \in S^h$.

Subtracting $(\mathcal{G}\frac{\partial u_i^h}{\partial t},\eta)$ from both sides of (3.3.2), substituting $\eta = e_i^h \in S^h$ into the resulting equation, and noting that $e_i = e_i^A + e_i^h$ and (3.2.48) we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_i\|_{-1}^2 + \gamma|e_i|_1^2 &= \gamma(\nabla e_i, \nabla e_i^A) + (\mathcal{G}\frac{\partial e_i}{\partial t}, e_i^A) + ((\mathcal{G}^h - \mathcal{G})\frac{\partial u_i^h}{\partial t}, e_i^h)\\ &\quad + \big(\phi(u_i^h), (I - ⨍\,)e_i^h\big)^h - \big(\phi(u_i), (I - ⨍\,)e_i^h\big)\\ &\quad + 2D\big(\big(\Psi_i(u_1^h, u_2^h), (I - ⨍\,)e_i^h\big)^h - \big(\Psi_i(u_1, u_2), (I - ⨍\,)e_i^h\big)\big)\\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.3.3}$$

where I_j correspond to the jth line of the terms in the right hand side.

Now we bound each line in turn. Write $I_1 = I_{1,1} + I_{1,2} + I_{1,3}$, where each $I_{1,k}$ denotes the kth term of I_1. Hence by the Cauchy-Schwarz and the Young inequality (2.1.4) we obtain

$$I_{1,1} \le \gamma|\nabla e_i|_0|\nabla e_i^A|_0 \le \gamma\big(\frac{1}{2\epsilon}|e_i|_1^2 + \frac{\epsilon}{2}|e_i^A|_1^2\big) = \frac{\gamma}{8}|e_i|_1^2 + C|e_i^A|_1^2. \tag{3.3.4}$$

Using the Cauchy-Schwarz inequality, in conjunction with the Poincaré inequality (2.1.2), and (2.1.3) we have

$$I_{1,2} \le \left|\mathcal{G}\frac{\partial e_i}{\partial t}\right|_0 |e_i^A|_0 \le C_p\left|\mathcal{G}\frac{\partial e_i}{\partial t}\right|_1 |e_i^A|_0 = C_p\left\|\frac{\partial e_i}{\partial t}\right\|_{-1} |e_i^A|_0. \tag{3.3.5}$$

Noting the Cauchy-Schwarz inequality, together with the Young inequality (2.1.4), and the Poincaré inequality (2.1.2) we can express $I_{1,3}$ as

$$\begin{aligned} I_{1,3} &\le \left|(\mathcal{G}^h - \mathcal{G})\frac{\partial u_i^h}{\partial t}\right|_0 |e_i^h|_0\\ &\le \frac{1}{2\epsilon}|e_i^h|_0^2 + \frac{\epsilon}{2}\left|(\mathcal{G}^h - \mathcal{G})\frac{\partial u_i^h}{\partial t}\right|_0^2\\ &\le \frac{1}{\epsilon}|e_i|_0^2 + \frac{1}{\epsilon}|e_i^A|_0^2 + \frac{\epsilon}{2}\left|(\mathcal{G}^h - \mathcal{G})\frac{\partial u_i^h}{\partial t}\right|_0^2\\ &\le \frac{C}{\epsilon}|e_i|_1^2 + \frac{1}{\epsilon}|e_i^A|_0^2 + \frac{\epsilon}{2}\left|(\mathcal{G}^h - \mathcal{G})\frac{\partial u_i^h}{\partial t}\right|_0^2\\ &\le \frac{\gamma}{8}|e_i|_1^2 + C|e_i^A|_0^2 + C\left|(\mathcal{G}^h - \mathcal{G})\frac{\partial u_i^h}{\partial t}\right|_0^2. \end{aligned} \tag{3.3.6}$$

To bound I_2, rewrite it as

$$I_2 = I_{2,1} + I_{2,2}, \tag{3.3.7}$$

where

$$I_{2,1} = \left((u_i^h)^3, (I - ⨍\,)e_i^h\right)^h - \left((u_i)^3, (I - ⨍\,)e_i^h\right), \tag{3.3.8}$$

$$I_{2,2} = \left(u_i, (I - ⨍\,)e_i^h\right) - \left(u_i^h, (I - ⨍\,)e_i^h\right)^h. \tag{3.3.9}$$

Adding and subtracting $\left(u_i^h, (I - ⨍\,)e_i^h\right)$, we can rewrite $I_{2,2}$ as

$$I_{2,2} \leq \left|\left(u_i - u_i^h, (I - ⨍\,)e_i^h\right)\right| + \left|\left(u_i^h, (I - ⨍\,)e_i^h\right) - \left(u_i^h, (I - ⨍\,)e_i^h\right)^h\right|.$$

Notice that by the Poincaré inequality (2.1.2) we have

$$\begin{aligned}
|(I - ⨍\,)e_i^h|_0 &\leq C_P\left(|(I - ⨍\,)e_i^h|_1 + |((I - ⨍\,)e_i^h, 1)|\right) \\
&= C_P\left(|e_i^h|_1 + |(e_i^h, 1) - (⨍\, e_i^h, 1)|\right) \\
&= C_P\left(|e_i^h|_1 + |(e_i^h, 1) - ⨍\, e_i^h(1, 1)|\right) \\
&= C_P\left(|e_i^h|_1 + |(e_i^h, 1) - |\Omega|\frac{1}{|\Omega|}(e_i^h, 1)|\right) \\
&= C_P|e_i^h|_1.
\end{aligned} \tag{3.3.10}$$

It follows by the definition of the norm in H^1 that

$$\|(I - ⨍\,)e_i^h\|_1 = \left(|(I - ⨍\,)e_i^h|_0^2 + |(I - ⨍\,)e_i^h|_1^2\right)^{\frac{1}{2}} \leq C|e_i^h|_1. \tag{3.3.11}$$

Noting the Cauchy-Schwarz inequality, (3.1.2b), (3.3.10), (3.3.11), the Young inequality (2.1.4), (3.2.8a), (2.1.5), and reapplying the Young inequality (2.1.4) we obtain

$$
\begin{aligned}
I_{2,2} &\le |e_i|_0 \; |(I - ⨍ \,)e_i^h|_0 + Ch^2 \|u_i^h\|_1 \; \|(I - ⨍ \,)e_i^h\|_1 \\
&\le C_P |e_i|_0 \; |e_i^h|_1 + Ch^2 \|u_i^h\|_1 \; |e_i^h|_1 \\
&\le C_P |e_i|_0^2 + \frac{\epsilon}{2}|e_i^h|_1^2 + Ch^4 \|u_i^h\|_1^2 + \frac{\epsilon}{2}|e_i^h|_1^2 \\
&\le C|e_i|_0^2 + \epsilon|e_i^h|_1^2 + Ch^4 \\
&\le C\|e_i\|_{-1}|e_i|_1 + 2\epsilon|e_i|_1^2 + 2\epsilon|e_i^A|_1^2 + Ch^4 \\
&\le C\|e_i\|_{-1}^2 + 4\epsilon|e_i|_1^2 + 2\epsilon|e_i^A|_1^2 + Ch^4 \\
&\le C\|e_i\|_{-1}^2 + \frac{\gamma}{8}|e_i|_1^2 + C|e_i^A|_1^2 + Ch^4. \qquad (3.3.12)
\end{aligned}
$$

To bound $I_{2,1}$ we add and subtract $\big((u_i^h)^3, (I - ⨍ \,)e_i^h\big)$ and note (2.2.55). This yields

$$
\begin{aligned}
I_{2,1} &= \big((u_i^h)^3, (I - ⨍ \,)e_i^h\big)^h - \big((u_i)^3, (I - ⨍ \,)e_i^h\big) \\
&= \big((u_i^h)^3, (I - ⨍ \,)e_i^h\big)^h - \big((u_i^h)^3, (I - ⨍ \,)e_i^h\big) + \big((u_i^h)^3 - (u_i)^3, (I - ⨍ \,)e_i^h\big) \\
&= I_{2,1,1} + I_{2,1,2}.
\end{aligned}
$$

Hence noting (3.1.43), (3.2.22), (3.3.11) and the Young inequality (2.1.4) we obtain

$$
\begin{aligned}
I_{2,1,1} &= \big((u_i^h)^3, (I - ⨍ \,)e_i^h\big)^h - \big((u_i^h)^3, (I - ⨍ \,)e_i^h\big) \\
&\le Ch^{2-d/3} \|u_i^h\|_1^3 \; \|(I - ⨍ \,)e_i^h\|_1 \\
&\le Ch^{2-d/3} |e_i^h|_1 \\
&\le Ch^{4-2d/3} + \frac{1}{\epsilon}|e_i^h|_1^2 \\
&\le Ch^{4-2d/3} + \frac{\gamma}{8}|e_i|_1^2 + C|e_i^A|_1 \qquad (3.3.13)
\end{aligned}
$$

Noting (2.1.9), (3.3.11), (3.2.8a), (2.1.5), and the Young inequality (2.1.4) we have

$$
\begin{aligned}
I_{2,1,2} &\le |((u_i)^3 - (u_i^h)^3, (I - ⨍\,)e_i^h)|, \\
&= |((u_i - u_i^h)(u_i^2 + u_i u_i^h + (u_i^h)^2), (I - ⨍\,)e_i^h)|, \\
&\le C|u_i - u_i^h|_0 \, \|(I - ⨍\,)e_i^h\|_1 (\|u_i\|_1^2 + \|u_i\|_1 \, \|u_i^h\|_1 + \|u_i^h\|_1^2), \\
&\le C|e_i|_0^2 + \frac{1}{2\epsilon}|e_i^h|_1^2, \\
&\le C\|e_i\|_{-1} \, |e_i|_1 + \frac{1}{2\epsilon}|e_i^h|_1^2, \\
&\le C\|e_i\|_{-1}^2 + \frac{2}{\epsilon}|e_i|_1^2 + C|e_i^A|_1^2, \\
&= C\|e_i\|_{-1}^2 + \frac{\gamma}{8}|e_i|_1^2 + C|e_i^A|_1^2.
\end{aligned} \tag{3.3.14}
$$

Now we bound the term I_3. To do so we rewrite I_3 for $i = 1$ as follows

$$
\begin{aligned}
&\big(\Psi_1(u_1^h, u_2^h), (I - ⨍\,)e_1^h\big)^h - \big(\Psi_1(u_1, u_2), (I - ⨍\,)e_1^h\big) \\
&\quad = \big((u_1^h + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big)^h - \big((u_1 + 1)(u_2 + 1)^2, (I - ⨍\,)e_1^h\big) \\
&\quad = \big((u_1^h + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big)^h - \big((u_1^h + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big) \\
&\qquad + \big((u_1^h + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big) - \big((u_1 + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big) \\
&\qquad + \big((u_1 + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big) - \big((u_1 + 1)(u_2 + 1)^2, (I - ⨍\,)e_1^h\big) \\
&\quad = I_{3,1} + I_{3,2} + I_{3,3}.
\end{aligned} \tag{3.3.15}
$$

Noting (3.1.43), (3.3.11), (3.2.8a) and the Young inequality (2.1.4) we obtain

$$
\begin{aligned}
I_{3,1} &\le \big|\big((u_1^h + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big)^h - \big((u_1^h + 1)(u_2^h + 1)^2, (I - ⨍\,)e_1^h\big)\big| \\
&\le Ch^{2-d/3}\|u_1^h + 1\|_1 \, \|u_2^h + 1\|_1^2 \, \|(I - ⨍\,)e_1^h\|_1 \\
&\le Ch^{2-d/3} \, |e_1^h|_1 \\
&\le Ch^{4-2d/3} + \epsilon|e_1^h|_1^2 \\
&\le Ch^{4-2d/3} + 2\epsilon|e_1|_1^2 + C|e_1^A|_1^2.
\end{aligned} \tag{3.3.16}
$$

Using (2.1.9), (3.2.8a), (3.3.11), the Young inequality (2.1.4), (2.1.5) and reapplying the Young inequality (2.1.4) we have

$$\begin{aligned}
I_{3,2} &= \left((u_1^h - u_1)(u_2^h + 1)^2, (I - \fint\ )e_1^h\right) \\
&\leq C|u_1^h - u_1|_0 \, \|u_2^h + 1\|_1^2 \, \|(I - \fint\ )e_1^h\|_1 \\
&\leq C|e_1|_0 \, |e_1^h|_1 \\
&\leq C|e_1|_0^2 + \epsilon |e_1^h|_1^2 \\
&\leq C\|e_1\|_{-1} \, |e_1|_1 + 2\epsilon |e_1|_1^2 + C|e_1^A|_1^2 \\
&\leq C\|e_1\|_{-1}^2 + 4\epsilon |e_1|_1^2 + C|e_1^A|_1^2.
\end{aligned} \tag{3.3.17}$$

Using the same technique we obtain

$$\begin{aligned}
I_{3,3} &= \left((u_1 + 1)((u_2^h + 1)^2 - (u_2 + 1)^2), (I - \fint\ )e_1^h\right) \\
&\leq C\|e_2\|_{-1}^2 + 2\epsilon |e_1|_1^2 + C|e_1^A|_1^2.
\end{aligned} \tag{3.3.18}$$

On substituting (3.3.16–3.3.18) into (3.3.15) and setting $\epsilon = \gamma / \, 128D$ we obtain the bound I_3 for $i = 1$

$$\begin{aligned}
&\left(\Psi_1(u_1^h, u_2^h), (I - \fint\ )e_1^h\right)^h - \left(\Psi_1(u_1, u_2), (I - \fint\ )e_1^h\right) \\
&\qquad \leq Ch^{4-2d/3} + 8\epsilon |e_1|_1^2 + C|e_1^A|_1^2 + C\|e_1\|_{-1}^2 + C\|e_2\|_{-1}^2 \\
&\qquad \leq Ch^{4-2d/3} + \frac{\gamma}{16D}|e_1|_1^2 + C|e_1^A|_1^2 + C\|e_1\|_{-1}^2 + C\|e_2\|_{-1}^2.
\end{aligned} \tag{3.3.19}$$

In the same way we have the bound I_3 for $i = 2$

$$\begin{aligned}
&\left(\Psi_2(u_1^h, u_2^h), (I - \fint\ )e_2^h\right)^h - \left(\Psi_2(u_1, u_2), (I - \fint\ )e_2^h\right) \\
&\qquad \leq Ch^{4-2d/3} + 8\epsilon |e_2|_1^2 + C|e_2^A|_1^2 + C\|e_1\|_{-1}^2 + C\|e_2\|_{-1}^2 \\
&\qquad \leq Ch^{4-2d/3} + \frac{\gamma}{16D}|e_2|_1^2 + C|e_2^A|_1^2 + C\|e_1\|_{-1}^2 + C\|e_2\|_{-1}^2.
\end{aligned} \tag{3.3.20}$$

On substituting (3.3.4–3.3.6), (3.3.12–3.3.14) into (3.3.3) simplifying, summing for $i = 1, 2$, and substituting (3.3.19)–(3.3.20) into the resulting equation we obtain *a.e.*

$$\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|e_1\|_{-1}^2+\frac{\mathrm{d}}{\mathrm{d}t}\|e_2\|_{-1}^2\right)+\frac{\gamma}{4}\left(|e_1|_1^2+|e_2|_1^2\right)$$
$$\leq C\big(\|e_1\|_{-1}^2+\|e_2\|_{-1}^2+|e_1^A|_1^2+|e_2^A|_1^2$$
$$+\left\|\frac{\partial e_1}{\partial t}\right\|_{-1}|e_1^A|_0+\left\|\frac{\partial e_2}{\partial t}\right\|_{-1}|e_2^A|_0+|e_1^A|_0^2+|e_2^A|_0^2$$
$$+\left|(\mathcal{G}^h-\mathcal{G})\frac{\partial u_1^h}{\partial t}\right|_0^2+\left|(\mathcal{G}^h-\mathcal{G})\frac{\partial u_2^h}{\partial t}\right|_0^2+h^2\big). \tag{3.3.21}$$

Integrating over $t\in(0,T)$, using a Grönwall inequality and rearranging the terms we obtain

$$\left(\|e_1(T)\|_{-1}^2+\|e_2(T)\|_{-1}^2\right)+\int_0^T\left(|e_1|_1^2+|e_2|_1^2\right)ds$$
$$\leq C\left(\|e_1(0)\|_{-1}^2+\|e_2(0)\|_{-1}^2\right)+C\int_0^T\Bigg(\|e_1^A\|_1^2+\|e_2^A\|_1^2$$
$$+\left\|\frac{\partial e_1}{\partial t}\right\|_{-1}\|e_1^A\|_1+\left\|\frac{\partial e_2}{\partial t}\right\|_{-1}\|e_2^A\|_1+\left|(\mathcal{G}^h-\mathcal{G})\frac{\partial u_1^h}{\partial t}\right|_0^2$$
$$+\left|(\mathcal{G}^h-\mathcal{G})\frac{\partial u_2^h}{\partial t}\right|_0^2+h^2\Bigg)ds.$$

Hence we obtain, for $d=1,2,3$,

$$\|e_1\|_{L^\infty(0,T;(H^1(\Omega))')}^2+\|e_2\|_{L^\infty(0,T;(H^1(\Omega))')}^2+\|e_1\|_{L^2(0,T;H^1(\Omega))}^2+\|e_2\|_{L^2(0,T;H^1(\Omega))}^2$$
$$\leq C\Bigg(\|(I-P^h)u_1^0\|_{-1}^2+\|(I-P^h)u_2^0\|_{-1}^2+\|e_1^A\|_{L^2(0,T;H^1(\Omega))}^2$$
$$+\|e_2^A\|_{L^2(0,T;H^1(\Omega))}^2+\left\|\frac{\partial e_1}{\partial t}\right\|_{L^2(0,T;(H^1(\Omega))')}\|e_1^A\|_{L^2(0,T;L^2(\Omega))}$$
$$+\left\|\frac{\partial e_2}{\partial t}\right\|_{L^2(0,T;(H^1(\Omega))')}\|e_2^A\|_{L^2(0,T;L^2(\Omega))}+\left\|(\mathcal{G}^h-\mathcal{G})\frac{\partial u_1^h}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}^2$$
$$+\left\|(\mathcal{G}^h-\mathcal{G})\frac{\partial u_2^h}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}^2\Bigg)+C(T)h^2\leq Ch^2,$$

where we have noted the Poincaré inequality (2.1.2), (3.1.12), (3.1.2c), (2.3.2), (2.2.6a), (2.2.6b), (3.2.8a), (3.2.8b), and (3.1.8) to obtain the last inequality. This ends the proof. □

Chapter 4

A Fully Discrete Approximation

In this chapter we introduce a numerical scheme (Scheme 1) to solve the weak formulation we mentioned in Chapter 1. We discuss the existence and uniqueness of the solutions for the scheme. We also discuss stability and convergence of the solution to the continuous problem in the weak formulation. We briefly mention a second scheme (Scheme 2) and show existence, uniqueness, and stability. We do not discuss the convergence of Scheme 2.

4.1 Notation

We shall now describe a finite element method for the numerical solution of the weak formulation of (1.0.7a-1.0.7f).

As in (3.1.4) we introduce the discrete Green's operator in the presence of the numerical integration $\widehat{\mathcal{G}}^h : S_0^h \mapsto S_0^h$ such that

$$(\nabla \widehat{\mathcal{G}}^h v, \nabla \eta) = (v, \eta)^h \quad \forall \eta \in S^h, \tag{4.1.1}$$

where $S_0^h := \{\eta \in S^h : (\eta, 1)^h = 0\}$.

We define a norm on S_0^h as

$$\|\eta^h\|_{-h,h}^2 := |\widehat{\mathcal{G}}^h \eta^h|_1^2 = (\nabla \widehat{\mathcal{G}}^h \eta^h, \nabla \widehat{\mathcal{G}}^h \eta^h) = (\widehat{\mathcal{G}}^h \eta^h, \eta^h)^h = (\eta^h, \widehat{\mathcal{G}}^h \eta^h)^h. \tag{4.1.2}$$

We have the following analogue to (3.1.6) and (3.1.7) respectively, that is for all

$\alpha > 0$,

$$(v^h, v^h)^h \equiv (\nabla \widehat{\mathcal{G}}^h v^h, \nabla v^h) \leq \frac{1}{2\alpha} |\widehat{\mathcal{G}}^h v^h|_1^2 + \frac{\alpha}{2} |v^h|_1^2 \quad \forall v^h \in S_0^h, \tag{4.1.3}$$

and

$$\|v^h\|_{-h,h}^2 \leq C_P |\widehat{\mathcal{G}}^h v^h|_1 |v^h|_0. \tag{4.1.4}$$

For later purposes, we recall the following inequality for the discrete Green's operator (see Blowey and Elliott [10])

$$\|(\mathcal{G}^h - \widehat{\mathcal{G}}^h) v^h\|_1 \leq C h^2 \|v^h\|_1 \quad \forall v^h \in V^h. \tag{4.1.5}$$

We also have the analogue of the inequalities (3.1.11), (see Barrett and Blowey [2]), as follows

$$h^2 |v^h|_1 \leq C_1 h |v^h|_h \leq C_2 |\widehat{\mathcal{G}}^h v^h|_1 \leq C_3 |\mathcal{G}^h v^h|_1 \leq C_4 |\widehat{\mathcal{G}}^h v^h|_1 \quad \forall v^h \in V^h. \tag{4.1.6}$$

The first inequality on the left is the inverse inequality on noting (3.1.2b). The second follows from the first and (4.1.3). The third and fourth follows (4.1.5) on noting the first two inequalities in (3.1.11) and (4.1.6) respectively.

In addition we have the analogue of Lemma 3.1.3 as follows:

Lemma 4.1.1 Let $v^h \in S^h$ and $d = 1, 2, 3$. Then

$$|v^h|_{0,4}^2 \leq C \|v^h\|_1^{\frac{7}{4}} \|v^h\|_{-h,h}^{\frac{1}{4}}. \tag{4.1.7}$$

Proof. It follows from (2.1.8), (3.1.2a) and (4.1.3). □

4.2 Scheme 1

4.2.1 Existence and Uniqueness

Given N, a positive integer, let $\Delta t = T/N$ denote a fixed time step, and $t^k = k\Delta t$ where $k = 0, \dots, N$. We focus our attention on approximating $(\mathbf{P})$ by the discrete scheme defined as follows:

$(\mathbf{P}_1^{h,\Delta t})$ Given U_1^0, U_2^0, find $\{U_1^n, U_2^n, W_1^n, W_2^n\} \in S^h \times S^h \times S^h \times S^h$, for $n = 1, \ldots, N$, such that $\forall \eta \in S^h$

$$(\frac{U_1^n - U_1^{n-1}}{\Delta t}, \eta)^h = -(\nabla W_1^n, \nabla \eta), \tag{4.2.1a}$$

$$(W_1^n, \eta)^h = (F_1(U_1^n, U_2^n), \eta)^h + \gamma(\nabla U_1^n, \nabla \eta), \tag{4.2.1b}$$

$$U_1^0 = P^h u_1^0, \tag{4.2.1c}$$

and

$$(\frac{U_2^n - U_2^{n-1}}{\Delta t}, \eta)^h = -(\nabla W_2^n, \nabla \eta), \tag{4.2.1d}$$

$$(W_2^n, \eta)^h = (F_2(U_1^n, U_2^n), \eta)^h + \gamma(\nabla U_2^n, \nabla \eta), \tag{4.2.1e}$$

$$U_2^0 = P^h u_2^0, \tag{4.2.1f}$$

where

$$F_1(U_1^n, U_2^n) = (U_1^n)^3 - U_1^{n-1} + D(U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + 1)^2, \tag{4.2.1g}$$

$$F_2(U_1^n, U_2^n) = (U_2^n)^3 - U_2^{n-1} + D(U_2^n + U_2^{n-1} + 2)(U_1^n + 1)^2. \tag{4.2.1h}$$

Note that (4.2.1a–c) is independent of U_2^n and (4.2.1d–f) is dependent on U_1^n.

Using (4.1.1), we can rewrite (4.2.1a,d) as

$$(\nabla(\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n), \nabla \eta) = 0 \quad \forall \eta \in S^h. \tag{4.2.2}$$

Taking $\eta = \widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n$ in (4.2.2), we have

$$|\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n|_1^2 = 0,$$

which implies

$$|\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n|_1 = |\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n - \fint W_i^n|_1 = 0,$$

where

$$⨍ W_i^n = \frac{1}{|\Omega|}(W_i^n, 1)^h = \frac{1}{|\Omega|}(F_i(U_1^n, U_2^n), 1)^h \quad \text{for } i = 1, 2. \tag{4.2.3}$$

Thus by the Poincaré inequality (3.1.3), we have

$$0 = |\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n - ⨍ W_i^n|_1 \geq C_P^{-1}|\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + W_i^n - ⨍ W_i^n|_h,$$

which implies

$$W_i^n = -\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}] + ⨍ W_i^n. \tag{4.2.4}$$

Noting (4.2.3) and (4.2.4), and

$$\begin{aligned}
(F_i(U_1^n, U_2^n), \eta)^h - &\frac{1}{|\Omega|}(F_i(U_1^n, U_2^n), 1)^h(1, \eta)^h \\
&= (F_i(U_1^n, U_2^n), \eta)^h - \frac{1}{|\Omega|}(1, \eta)^h(F_i(U_1^n, U_2^n),)^h \\
&= (F_i(U_1^n, U_2^n), \eta)^h - (F_i(U_1^n, U_2^n), (1, \frac{\eta}{|\Omega|})^h)^h \\
&= (F_i(U_1^n, U_2^n), \eta)^h - (F_i(U_1^n, U_2^n), ⨍ \eta)^h \\
&= (F_i(U_1^n, U_2^n), (I - ⨍)\eta)^h,
\end{aligned} \tag{4.2.5}$$

we can restate the problem $(\mathbf{P}_1^{h,\Delta t})$ as:

Given U_1^0, U_2^0, find $\{U_1^n, U_2^n\} \in S^h \times S^h$, for $n = 1, \ldots, N$ such that $\forall \eta \in S^h$

$$(\widehat{\mathcal{G}}^h[\frac{U_1^n - U_1^{n-1}}{\Delta t}], \eta)^h + \gamma(\nabla U_1^n, \nabla \eta) + (F_1(U_1^n, U_2^n), (I - ⨍)\eta)^h = 0, \tag{4.2.6a}$$

$$(\widehat{\mathcal{G}}^h[\frac{U_2^n - U_2^{n-1}}{\Delta t}], \eta)^h + \gamma(\nabla U_2^n, \nabla \eta) + (F_2(U_1^n, U_2^n), (I - ⨍)\eta)^h = 0. \tag{4.2.6b}$$

Theorem 4.2.1 Let the assumptions of Theorem 3.2.1 hold and $\Delta t > 0$. Then there exists a unique solution $\{U_1, U_2, W_1, W_2\}$ to $(\mathbf{P}_1^{h,\Delta t})$ such that the following stability bounds hold:

$$\max_{m=1,\dots,N}\Bigg\{\Delta t\sum_{n=1}^{m}\left(|\widehat{\mathcal{G}}^h[\frac{U_1^n-U_1^{n-1}}{\Delta t}]|_1^2+|\widehat{\mathcal{G}}^h[\frac{U_2^n-U_2^{n-1}}{\Delta t}]|_1^2\right)$$
$$+\frac{\gamma}{2}\Big(|U_1^m|_1^2+|U_2^m|_1^2+\sum_{n=1}^{m}(|U_1^n-U_1^{n-1}|_1^2+|U_2^n-U_2^{n-1}|_1^2)\Big)$$
$$+\frac{1}{2}\sum_{n=1}^{m}\left(|U_1^n-U_1^{n-1}|_h^2+|U_2^n-U_2^{n-1}|_h^2\right)$$
$$+\frac{1}{4}((U_1^m)^4-2(U_1^m)^2+(U_2^m)^4-2(U_2^m)^2,1)^h$$
$$+D((U_2^m+1)^2(U_1^m+1)^2,1)^h\Bigg\}\leq C, \tag{4.2.7}$$

$$\Delta t\sum_{n=1}^{N}||W_1^n||_1^2+\Delta t\sum_{n=1}^{N}||W_2^n||_1^2\leq C, \tag{4.2.8}$$

$$|F_1(U_1^n,U_2^n)|_h^2+|F_2(U_1^n,U_2^n)|_h^2\leq C. \tag{4.2.9}$$

Proof. Let

$$K_1^h=\{\eta_1\in S^h:(\eta_1,1)^h=m_1:=⨍\, u_1^0\}, \tag{4.2.10a}$$
$$K_2^h=\{\eta_2\in S^h:(\eta_2,1)^h=m_2:=⨍\, u_2^0\}. \tag{4.2.10b}$$

Consider the coupled variational problems

$$\min_{\eta_1\in K_1^h}\ \mathcal{E}_1^h(\eta_1),\qquad \min_{\eta_2\in K_2^h}\ \mathcal{E}_2^h(\eta_2),$$

where

$$\mathcal{E}_1^h(\eta_1)=\frac{1}{2}|\widehat{\mathcal{G}}^h[\frac{\eta_1-U_1^{n-1}}{\Delta t}]|_1^2+\frac{\gamma}{2}|\eta_1|_1^2+(\frac{1}{4}\eta_1^4,1)^h-(U_1^{n-1},\eta_1)^h$$
$$+D((U_2^{n-1}+1)^2,\frac{1}{2}\eta_1^2+(U_1^{n-1}+2)\eta_1)^h,$$
$$\mathcal{E}_2^h(\eta_2)=\frac{1}{2}|\widehat{\mathcal{G}}^h\frac{\eta_2-U_2^{n-1}}{\Delta t}|_1^2+\frac{\gamma}{2}|\eta_2|_1^2+(\frac{1}{4}\eta_2^4,1)^h-(U_2^{n-1},\eta_2)^h$$
$$+D((U_1^n+1)^2,\frac{1}{2}\eta_2^2+(U_2^{n-1}+2)\eta_2)^h.$$

Since $|\widehat{\mathcal{G}}^h[\frac{\eta_i - U_i^{n-1}}{\Delta t}]|_1^2 \geq 0$ for $i = 1, 2$, we have

$$\mathcal{E}_1^h(\eta_1) \geq \frac{\gamma}{2}|\eta_1|_1^2 + (\frac{1}{4}\eta_1^4, 1)^h - (U_1^{n-1}, \eta_1)^h + \frac{D}{2}((U_2^{n-1}+1)^2, \eta_1^2)^h$$
$$+ D((U_2^{n-1}+1)\eta_1, (U_1^{n-1}+2)(U_2^{n-1}+1))^h, \tag{4.2.11}$$
$$\mathcal{E}_2^h(\eta_2) \geq \frac{\gamma}{2}|\eta_2|_1^2 + (\frac{1}{4}\eta_2^4, 1)^h - (U_2^{n-1}, \eta_2)^h + \frac{D}{2}((U_1^{n}+1)^2, \eta_2^2)^h$$
$$+ D((U_1^{n}+1)\eta_2, (U_2^{n-1}+2)(U_1^{n}+1))^h. \tag{4.2.12}$$

By noting the Young inequality (2.1.4), for $p = 4, q = 4/3$ we are able to show

$$-(U_1^{n-1}, \eta_1)^h \geq -\frac{1}{4}(\eta_1^4, 1)^h - \frac{3}{4}\int_\Omega \pi^h[(U_1^{n-1})^{\frac{4}{3}}]dx, \tag{4.2.13}$$

and for $p = q = 2$, we obtain

$$((U_2^{n-1}+1)\eta_1, (U_1^{n-1}+2)(U_2^{n-1}+1))^h \geq -\frac{1}{2}|(U_2^{n-1}+1)\eta_1|_h^2$$
$$-\frac{1}{2}|(U_2^{n-1}+1)(U_1^{n-1}+2)|_h^2. \tag{4.2.14}$$

We also note that

$$((U_2^{n-1}+1)^2, \eta_1^2)^h = |(U_2^{n-1}+1)\eta_1|_h^2. \tag{4.2.15}$$

Similarly, we have

$$-(U_2^{n-1}, \eta_2)^h \geq -\frac{1}{4}(\eta_2^4, 1)^h - \frac{3}{4}\int_\Omega \pi^h[(U_2^{n-1})^{\frac{4}{3}}]dx, \tag{4.2.16}$$
$$((U_1^{n}+1)\eta_2, (U_1^{n}+1)(U_2^{n-1}+2))^h \geq -\frac{1}{2}|(U_1^{n}+1)\eta_2|_h^2$$
$$-\frac{1}{2}|(U_1^{n}+1)(U_2^{n-1}+2)|_h^2, \tag{4.2.17}$$
$$((U_1^{n}+1)^2, \eta_2^2)^h = |(U_1^{n}+1)\eta_2|_h^2. \tag{4.2.18}$$

On substituting (4.2.13–4.2.15) into (4.2.11) and (4.2.16–4.2.18) into (4.2.12), and simplifying, we arrive respectively at

$$\mathcal{E}_1^h(\eta_1) \geq \frac{\gamma}{2}|\eta_1|_1^2 - D|(U_2^{n-1}+1)(U_1^{n-1}+2)|_h^2 - \frac{3}{4}\int_\Omega \pi^h(U_1^{n-1})^{\frac{4}{3}}dx \tag{4.2.19}$$

$$\mathcal{E}_2^h(\eta_2) \geq \frac{\gamma}{2}|\eta_2|_1^2 - D|(U_1^{n}+1)(U_2^{n-1}+2)|_h^2 - \frac{3}{4}\int_\Omega \pi^h(U_2^{n-1})^{\frac{4}{3}}dx. \tag{4.2.20}$$

Thus $\mathcal{E}_1^h$ and $\mathcal{E}_2^h$ are bounded below in K_1^h and K_2^h respectively.

Now let $d_i = \inf_{K_i^h} \mathcal{E}_i^h(\eta_i)$ and $\{\eta_{i,n}\}$ be a minimising sequence of $\mathcal{E}_i^h$ in K_i^h, i.e, $\lim_{n\to\infty} \mathcal{E}_i^h(\eta_{i,n}) = d_i$. It follows from the above estimate and the discrete Poincaré inequality (3.1.3) that $\{\eta_{i,n}\} \in H^1(\Omega)$. Recalling that K_i^h are finite dimensional spaces, then there exist $U_i \in S^h$ and subsequences $\{\eta_{i,m}\}$ such that

$$\{\eta_{i,m}\} \to U_i \in S^h.$$

Since K_i^h are closed and $U_i \in K_i^h$ then the continuity of $\mathcal{E}_i^h$ give $\mathcal{E}_i^h(\eta_{i,n}) \to \mathcal{E}_i^h(U_i) = d_i$. As a consequence, there exist solutions U_i to the coupled variational problems. Such minimisers, which are critical points of $\mathcal{E}_1^h$ and $\mathcal{E}_2^h$, satisfy the Euler Lagrange equation of $\mathcal{E}_1^h$ and $\mathcal{E}_2^h$ given by (4.2.6a) and (4.2.6b) respectively, while $\{W_1, W_2\}$ exist from (4.2.4) and (4.2.3) .

To show the uniqueness, let $U_1^{n,1}, U_2^{n,1}$ and $U_1^{n,2}, U_2^{n,2}$ be two solutions of $(\mathbf{P}^{h,\Delta t})$ and set

$$Z_1 = U_1^{n,1} - U_1^{n,2}, \qquad Z_2 = U_2^{n,1} - U_2^{n,2}. \tag{4.2.21}$$

From (4.2.6a) we have, for $i = 1, 2$,

$$(\widehat{\mathcal{G}}^h[\frac{U_1^{n,i} - U_1^{n-1}}{\Delta t}], \eta)^h + \gamma(\nabla U_1^{n,i}, \nabla \eta) + ((U_1^{n,i})^3 - U_1^{n-1}, \eta - ⨍\, \eta)^h$$
$$+ D\big((U_1^{n,i} + U_1^{n-1} + 2)(U_2^{n-1}+1)^2, \eta - ⨍\, \eta\big)^h = 0. \tag{4.2.22}$$

Subtracting these equations and noting (4.2.21) yields

$$\frac{1}{\Delta t}(\widehat{\mathcal{G}}^h Z_1, \eta)^h + \gamma(\nabla Z_1, \nabla \eta) + ((U_1^{n,1})^3 - (U_1^{n,2})^3, \eta - ⨍\, \eta)^h$$
$$+ D((U_2^{n-1}+1)^2 Z_1, \eta - ⨍\, \eta)^h = 0. \tag{4.2.23}$$

Taking $\eta = Z_1$ in (4.2.23) and noting (4.1.2) and (4.2.21), we have

$$\frac{1}{\Delta t}|\widehat{\mathcal{G}}^h Z_1|_1^2 + \gamma |Z_1|_1^2 + ((U_1^{n,1})^3 - (U_1^{n,2})^3, U_1^{n,1} - U_1^{n,2})^h + D|(U_2^{n-1}+1)Z_1|_h^2 = 0.$$

From the convexity it follows that

$$r^3(s-r) \le \frac{1}{4}s^4 - \frac{1}{4}r^4, \tag{4.2.24}$$

which implies

$$(r^3 - s^3)(r-s) \ge 0. \tag{4.2.25}$$

Thus

$$\frac{1}{\Delta t}|\widehat{\mathcal{G}}^h Z_1|_1^2 + \gamma |Z_1|_1^2 + D|(U_2^{n-1}+1)Z_1|_h^2 \le 0, \tag{4.2.26}$$

and it follows from (3.1.3) that

$$|Z_1|_h^2 = 0, \quad \text{i.e.} \quad Z_1 \equiv 0.$$

In the same way, we are able to show that

$$Z_2 \equiv 0.$$

Thus we have shown the uniqueness.

Next we will show the stability bound (4.2.7). Substituting $\eta = \frac{U_1^n - U_1^{n-1}}{\Delta t} \in S_0^h$ into (4.2.6a), we have

$$(\widehat{\mathcal{G}}^h[\frac{U_1^n - U_1^{n-1}}{\Delta t}], [\frac{U_1^n - U_1^{n-1}}{\Delta t}])^h + \gamma(\nabla U_1^n, \nabla[\frac{U_1^n - U_1^{n-1}}{\Delta t}]) \\ + (F_1(U_1^n, U_2^n), [\frac{U_1^n - U_1^{n-1}}{\Delta t}])^h = 0.$$

Noting (4.1.2) and the identity

$$2a(a-b) = a^2 - b^2 + (a-b)^2, \tag{4.2.27}$$

we have

$$\begin{aligned}\Delta t|\widehat{\mathcal{G}}^h[\frac{U_1^n - U_1^{n-1}}{\Delta t}]|_1^2 + \frac{\gamma}{2}|U_1^n|_1^2 - \frac{\gamma}{2}|U_1^{n-1}|_1^2 + \frac{\gamma}{2}|U_1^n - U_1^{n-1}|_1^2 \\ =((U_1^n)^3, U_1^{n-1} - U_1^n)^h + (U_1^{n-1}, U_1^n - U_1^{n-1})^h \\ + D((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + 1)^2, U_1^{n-1} - U_1^n)^h.\end{aligned} \tag{4.2.28a}$$

In the same way, from the equation (4.2.6b), we obtain

$$\begin{aligned}\Delta t|\widehat{\mathcal{G}}^h[\frac{U_2^n - U_2^{n-1}}{\Delta t}]|_1^2 + \frac{\gamma}{2}|U_2^n|_1^2 - \frac{\gamma}{2}|U_2^{n-1}|_1^2 + \frac{\gamma}{2}|U_2^n - U_2^{n-1}|_1^2 \\ =((U_2^n)^3, U_2^{n-1} - U_2^n)^h + (U_2^{n-1}, U_2^n - U_2^{n-1})^h \\ + D((U_2^n + U_2^{n-1} + 2)(U_1^n + 1)^2, U_2^{n-1} - U_2^n)^h.\end{aligned} \tag{4.2.28b}$$

Now let us consider the right hand side terms of (4.2.28a). By noting the inequality (4.2.24) we obtain

$$((U_1^n)^3, U_1^{n-1} - U_1^n)^h \leq \frac{1}{4}((U_1^{n-1})^4 - (U_1^n)^4, 1)^h. \tag{4.2.29}$$

On noting (4.2.27), we have

$$(U_1^{n-1}, U_1^n - U_1^{n-1})^h = \frac{1}{2}\left(-|U_1^{n-1}|_h^2 + |U_1^n|_h^2 - |U_1^n - U_1^{n-1}|_h^2\right). \tag{4.2.30}$$

The last term on the right hand side (4.2.28a) can be expressed as

$$\begin{aligned}((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + 1)^2, U_1^{n-1} - U_1^n)^h \\ =((U_1^{n-1} + 1)^2(U_2^{n-1} + 1)^2 - (U_1^n + 1)^2(U_2^{n-1} + 1)^2, 1)^h.\end{aligned} \tag{4.2.31}$$

In the same way, from the right hand side (4.2.28b), we have

$$((U_2^n)^3, U_2^{n-1} - U_2^n)^h \leq \frac{1}{4}((U_2^{n-1})^4 - (U_2^n)^4, 1)^h, \tag{4.2.32}$$

$$(U_2^{n-1}, U_2^n - U_2^{n-1})^h = \frac{1}{2}\big(- |U_2^{n-1}|_h^2 + |U_2^n|_h^2 - |U_2^n - U_2^{n-1}|_h^2\big), \tag{4.2.33}$$

and

$$((U_2^n + U_2^{n-1} + 2)(U_1^n + 1)^2, U_2^{n-1} - U_2^n)^h$$
$$= ((U_2^{n-1} + 1)^2(U_1^n + 1)^2 - (U_2^n + 1)^2(U_1^n + 1)^2, 1)^h. \tag{4.2.34}$$

Substituting (4.2.29–4.2.31) into (4.2.28a) and (4.2.32–4.2.34) into (4.2.28b), and adding the resulting equations, we have

$$\begin{aligned}
&\Delta t(|\widehat{\mathcal{G}}^h[\frac{U_1^n - U_1^{n-1}}{\Delta t}]|_1^2 + |\widehat{\mathcal{G}}^h[\frac{U_2^n - U_2^{n-1}}{\Delta t}]|_1^2) \\
&\quad + \frac{\gamma}{2}\left((|U_1^n|_1^2 - |U_1^{n-1}|_1^2) + (|U_2^n|_1^2 - |U_2^{n-1}|_1^2)\right) \\
&\quad + \frac{\gamma}{2}\left(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2\right) \\
&\quad + \frac{1}{2}\left(|U_1^{n-1}|_h^2 - |U_1^n|_h^2 + |U_2^{n-1}|_h^2 - |U_2^n|_h^2\right) \\
&\quad + \frac{1}{2}\left(|U_1^n - U_1^{n-1}|_h^2 + |U_2^n - U_2^{n-1}|_h^2\right) \\
&\quad + \frac{1}{4}((U_1^n)^4 - (U_1^{n-1})^4, 1)^h + \frac{1}{4}((U_2^n)^4 - (U_2^{n-1})^4, 1)^h \\
&\quad + D((U_1^n + 1)^2(U_2^n + 1)^2 - (U_2^{n-1} + 1)^2(U_1^{n-1} + 1)^2, 1)^h \le 0.
\end{aligned} \tag{4.2.35}$$

Summing the above equation over $n = 1, \ldots, m$ for $m \le N$ and rearranging the terms yield, for $d = 1, 2, 3$,

$$\begin{aligned}
&\max_{m=1,\ldots,N}\Bigg\{\Delta t \sum_{n=1}^{m}\left(|\widehat{\mathcal{G}}^h[\frac{U_1^n - U_1^{n-1}}{\Delta t}]|_1^2 + |\widehat{\mathcal{G}}^h[\frac{U_2^n - U_2^{n-1}}{\Delta t}]|_1^2\right) \\
&\qquad + \frac{\gamma}{2}\Big(|U_1^m|_1^2 + |U_2^m|_1^2 + \sum_{n=1}^{m}(|U_1^n - U_1^{n-1}|_1^2 + |U_2^n - U_2^{n-1}|_1^2)\Big) \\
&\qquad + \frac{1}{2}\sum_{n=1}^{m}\left(|U_1^n - U_1^{n-1}|_h^2 + |U_2^n - U_2^{n-1}|_h^2\right) \\
&\qquad + \frac{1}{4}((U_1^m)^4 - 2(U_1^m)^2 + (U_2^m)^4 - (U_2^m)^2, 1)^h \\
&\qquad + D((U_1^m + 1)^2(U_2^m + 1)^2, 1)^h\Bigg\} \\
&\qquad \le \frac{\gamma}{2}(|U_1^0|_1^2 + |U_2^0|_1^2) + \frac{1}{4}((U_1^0)^4 - 2(U_1^0)^2 + (U_2^0)^4 - 2(U_2^0)^2, 1)^h \\
&\qquad\quad + D((U_1^0 + 1)^2(U_2^0 + 1)^2, 1)^h \le C,
\end{aligned} \tag{4.2.36}$$

where we have noted the Young inequality (2.1.4), (3.1.40), (4.2.1c), (4.2.1f), (3.1.14) and the condition on u_i^0 for $i = 1, 2$, to obtain the last inequality.

To show (4.2.8), consider the Poincaré inequality (3.1.3), that is

$$|W_i^n|_0 \leq C_P(|W_i^n|_1 + |(W_i^n, 1)^h|).$$

The equation (4.2.3) and the Young inequality (2.1.4) with $p = q = 2$ yield

$$|W_i^n|_0^2 \leq C(|W_i^n|_1^2 + |\Omega|^2 |⨍ W_i^n|^2). \tag{4.2.37}$$

It follows that

$$||W_i^n||_1^2 \leq C(|W_i^n|_1^2 + |\Omega|^2 |⨍ W_i^n|^2). \tag{4.2.38}$$

Multiplying both sides of (4.2.38) with Δt and summing over n we have

$$\Delta t \sum_{n=1}^{N} ||W_i^n||_1^2 \leq C\Delta t \sum_{n=1}^{N} |W_i^n|_1^2 + C\Delta t |\Omega|^2 \sum_{n=1}^{N} |⨍ W_i^n|^2. \tag{4.2.39}$$

Recalling (4.2.4) and (4.2.7), it is enough to bound $|⨍ W_i^n|$ to conclude $||W_i^n||_1$ is bounded.

On noting (4.2.3) we have for $i = 1$

$$\begin{aligned}
|⨍ W_1^n| = &\frac{1}{|\Omega|}\left| \int_\Omega \pi^h \left[(U_1^n)^3 - U_1^n + D(U_1^n + U_1^{n-1} + 2)(U_j^{n-1} + 1)^2 \right] dx \right| \\
\leq &\frac{1}{|\Omega|}\left(\left| \int_\Omega \pi^h[(U_1^n)^3] dx \right| + \left| \int_\Omega \pi^h[U_1^n] dx \right| \right. \\
&\left. + \frac{D}{2}\left| \int_\Omega \pi^h[(U_1^n + 1)^2 + (U_1^{n-1} + 1)^2 + 2(U_2^{n-1} + 1)^4] dx \right| \right) \\
\leq &\frac{1}{|\Omega|}\left[\left| \int_\Omega \pi^h[(U_1^n)^3] dx \right| + \left| \int_\Omega \pi^h[U_1^n] dx \right| + \frac{D}{2}\left(\left| \int_\Omega \pi^h[(U_1^n + 1)^2] dx \right| \right.\right. \\
&\left.\left. + \left| \int_\Omega \pi^h[(U_1^{n-1} + 1)^2] dx \right| + 2\left| \int_\Omega \pi^h[(U_2^{n-1} + 1)^4] dx \right| \right) \right].
\end{aligned}$$

The inequality (3.1.40), (3.1.13) together with the conservation of mass, and (4.2.36) give

$$|⨍ W_1^n| \leq C \quad \text{for } d = 1, 2, 3. \tag{4.2.40a}$$

Using the same technique we are able to show

$$|\fint W_2^n| \leq C \quad \text{for } d = 1, 2, 3. \tag{4.2.40b}$$

Substituting (4.2.40a–b) into (4.2.39) yields the desired result (4.2.8).

Now we bound $|F_i(U_1^n, U_2^n)|_h^2$. To do so consider (3.1.2a), that is

$$\begin{aligned}|F_1(U_1^n, U_2^n)|_h^2 &= \int_\Omega \pi^h\Big[\big((U_1^n)^3 - U_1^{n-1} + D(U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + 1)^2\big)^2\Big]\,dx \\ &\leq \int_\Omega \Big(\pi^h\big[(U_1^n)^6\big] + \pi^h\big[(U_1^{n-1})^2\big] \\ &\qquad + \pi^h\big[(U_1^n + U_1^{n-1} + 2)^4(U_2^{n-1} + 1)^4\big]\Big)\,dx.\end{aligned} \tag{4.2.41}$$

Noting the Young inequality (2.1.4), and (3.1.40) we have for $d = 1, 2, 3$,

$$|F_1(U_1^n, U_2^n)|_h^2 \leq C\big(\|U_1^n\|_1^6 + \|U_1^{n-1}\|_1^6 + \|U_2^{n-1}\|_1^6\big) < C.$$

Similarly

$$|F_2(U_1^n, U_2^n)|_h^2 \leq C\big(\|U_2^n\|_1^6 + \|U_2^{n-1}\|_1^6 + \|U_1^n\|_1^6\big) < C.$$

This ends the proof of the theorem. □

Note that on defining

$$\mathcal{E}^h(U_1, U_2) := \int_\Omega \Big[\pi^h\psi(U_1) + \frac{\gamma}{2}|\nabla U_1|^2 + \pi^h\psi(U_2) + \frac{\gamma}{2}|\nabla U_2|^2 + 2D\pi^h\Psi(U_1, U_2)\Big]\,dx,$$

for $U_1, U_2 \in S^h$, the equation (4.2.35) shows that

$$\mathcal{E}^h(U_1^n, U_2^n) \leq \mathcal{E}^h(U_1^{n-1}, U_2^{n-1}),$$

that is $\mathcal{E}^h(\cdot, \cdot)$ is a discrete Lyapunov functional.

4.2.2 Convergence

To show the convergence of Scheme 1 introduced in this chapter to the weak form of the coupled pair of Cahn-Hilliard equations we follow closely the idea in Copetti and Elliott [16]. Let $\{U_1^n, U_2^n, W_1^n, W_2^n\}$ be the sequences resulting from (4.2.1a–4.2.1f). For $t \in (t^{n-1}, t^n)$, $1 \le n \le N$, we define the piecewise constant sequence in time, for $i = 1, 2$, by

$$u_{\Delta t,i}^h(t) = U_i^n, \qquad w_{\Delta t,i}^h(t) = W_i^n, \qquad F_{i,\Delta t}^h(t) = \pi^h F_i(U_1^n, U_2^n),$$
$$\mu_{\Delta t}(t) = \mu(t^{n-1}) \quad \text{for} \;\; \mu \in C^\infty(0,T),$$

and denote by $\bar{u}_{\Delta t,i}^h$ and $\bar{\mu}_{\Delta t}(t^n)$ the piecewise linear continuous functions on $[0,T]$ defined respectively by

$$\begin{aligned}
\bar{u}_{\Delta t,i}^h(t^n) &= U_i^n && \text{for } n = 0, \ldots, N, \\
\bar{\mu}_{\Delta t}(t^n) &= \mu(t^n) && \text{for } n = 0, \ldots, N-1, \\
\bar{\mu}_{\Delta t}(t^N) &= \mu(t^{N-1}).
\end{aligned}$$

Notice that Theorem 4.2.1 implies that

$$\begin{aligned}
&\|u_{\Delta t,1}^h\|_{L^\infty(0,T;H^1(\Omega))} + \|u_{\Delta t,2}^h\|_{L^\infty(0,T;H^1(\Omega))} \\
&\quad + \|\bar{u}_{\Delta t,1}^h\|_{L^\infty(0,T;H^1(\Omega))} + \|\bar{u}_{\Delta t,2}^h\|_{L^\infty(0,T;H^1(\Omega))} \\
&\quad + \|w_{\Delta t,1}^h\|_{L^2(0,T;H^1(\Omega))} + \|w_{\Delta t,2}^h\|_{L^2(0,T;H^1(\Omega))} \\
&\quad + \|F_{1,\Delta t}^h\|_{L^2(0,T;L^2(\Omega))} + \|F_{2,\Delta t}^h\|_{L^2(0,T;L^2(\Omega))} \\
&\quad + \left\|\frac{d}{dt}\bar{u}_{\Delta t,1}^h\right\|_{L^2(0,T;(H^1(\Omega))')} + \left\|\frac{d}{dt}\bar{u}_{\Delta t,2}^h\right\|_{L^2(0,T;(H^1(\Omega))')} \le C. \qquad (4.2.42)
\end{aligned}$$

Since $L^\infty(0,T;H^1(\Omega))$ is the dual space of $L^1(0,T;(H^1(\Omega))')$ (see Renardy and Rogers [33] page 378), which is separable, then there exist $u_i \in L^\infty(0,T;H^1(\Omega))$ and subsequences $\{u_{\Delta t,i}^h\}$, and $\{\bar{u}_{\Delta t,i}^h\} \in L^\infty(0,T;H^1(\Omega))$ such that

$$\begin{aligned}
u_{\Delta t,i}^h &\to u_i \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{weak-star}, \\
\bar{u}_{\Delta t,i}^h &\to u_i \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)) \quad \text{weak-star}.
\end{aligned}$$

Since $L^2(0,T;H^1(\Omega))$ is a reflexive Banach space (see Theorem 2.20.4 in Kufner *et.al.* [26]), then there exist w_i, $v_i \in L^2(0,T;H^1(\Omega))$ and subsequences $\{w^h_{\Delta t,i}\}$, $\{F^h_{i,\Delta t}\}$, such that

$$\begin{aligned} w^h_{\Delta t,i} &\to w_i \quad \text{in} \quad L^2(0,T;H^1(\Omega)) \quad \text{weakly,} \\ F^h_{i,\Delta t} &\to v_i \quad \text{in} \quad L^2(0,T;H^1(\Omega)) \quad \text{weakly.} \end{aligned}$$

Similarly, $L^2(0,T;(H^1(\Omega))')$ is a reflexive Hilbert space so there exist $d\bar{u}_i/dt \in L^2(0,T;(H^1(\Omega))')$ and subsequences $\{\bar{u}^h_{\Delta t,i}\} \in L^2(0,T;(H^1(\Omega))')$ such that

$$\frac{d}{dt}\bar{u}^h_{\Delta t,i} \to \frac{d}{dt}\bar{u}_i \quad \text{in} \quad L^2(0,T;(H^1(\Omega))') \quad \text{weakly.}$$

Also as $\Delta t \to 0$ we have

$$\begin{aligned} \bar{\mu}_{\Delta t} &\to \mu \quad \text{in} \quad L^2(0,T), \\ \frac{d}{dt}\bar{\mu}_{\Delta t} &\to \frac{d}{dt}\mu \quad \text{in} \quad L^2(0,T). \end{aligned}$$

Note that $H^1(\Omega)$ and $(H^1(\Omega))'$ are reflexive, and the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact. Hence the compactness theorem of Lions (see Theorem 5.1 in Lions [27] page 56) guarantees the existence of subsequence in $L^2(0,T,L^2(\Omega))$ such that

$$\bar{u}^h_{\Delta t,i} \to u_i \quad \text{in} \quad L^2(0,T,L^2(\Omega)). \tag{4.2.43}$$

Observing that

$$\begin{aligned} ||\bar{u}^h_{\Delta t,i} - u^h_{\Delta t,i}||^2_{L^2(0,T;H^1(\Omega))} &= \int_0^T ||\bar{u}^h_{\Delta t,i} - u^h_{\Delta t,i}||^2_1 dt \\ &= \sum_{n=1}^{N} \int_{(n-1)\Delta t}^{n\Delta t} ||\bar{u}^h_{\Delta t,i} - u^h_{\Delta t,i}||^2_1 dt \\ &\leq \Delta t \sum_{n=1}^{N} ||U^n_i - U^{n-1}_i||^2_1 \leq C\Delta t, \end{aligned}$$

where we have noted Theorem 4.2.1 to obtain the last inequality. This implies that

$$u^h_{\Delta t,i} \to u_i \quad \text{in} \quad L^2(0,T,H^1(\Omega)). \tag{4.2.44}$$

Let $\xi \in H^1(\Omega)$ be arbitrary. Setting $\eta = P_1^h\xi$ in (4.2.1a–b) and (4.2.1d–e) we have, for $i = 1, 2$,

$$\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}, P_1^h\xi\right)^h + (\nabla W_i^n, \nabla P_1^h\xi) = 0, \tag{4.2.45}$$

$$(W_i^n, P_1^h\xi)^h - (F_i(U_1^n, U_2^n), P_1^h\xi)^h - \gamma(\nabla U_i^n, \nabla P_1^h\xi) = 0. \tag{4.2.46}$$

Multiplying each equation by $\Delta t\mu(t^{n-1})$, we have

$$\Delta t\mu(t^{n-1})\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}, P_1^h\xi\right)^h + \Delta t\mu(t^{n-1})(\nabla W_i^n, \nabla P_1^h\xi) = 0,$$
$$\Delta t\mu(t^{n-1})\big((W_i^n, P_1^h\xi)^h - (F_i(U_1^n, U_2^n), P_1^h\xi)^h - \gamma(\nabla U_i^n, \nabla P_1^h\xi)\big) = 0.$$

Summing over n we obtain

$$\Delta t\sum_{n=1}^{N-1}\frac{\mu(t^n) - \mu(t^{n-1})}{\Delta t}(U_i^n, P_1^h\xi)^h + \mu(t^{N-1})(U_i^N, P_1^h\xi)^h$$
$$-\mu(t^0)(U_i^0, P_1^h\xi)^h + \Delta t\sum_{n=1}^{N}\mu(t^{n-1})(\nabla W_i^n, \nabla P_1^h\xi) = 0,$$
$$\Delta t\sum_{n=1}^{N}\mu(t^{n-1})[(W_i^n, P_1^h\xi)^h - (F_i(U_1^n, U_2^n), P_1^h\xi)^h - \gamma(\nabla U_i^n, \nabla P_1^h\xi)] = 0.$$

These equations are equivalent to

$$-\int_0^T (u^h_{\Delta t,i}, P_1^h\xi)^h \frac{d}{dt}\bar{\mu}_{\Delta t}(t)dt + \mu(t^{N-1})(U_i^N, P_1^h\xi)^h - \mu(t^0)(u_i^0, P_1^h\xi)^h$$
$$+\int_0^T \mu_{\Delta t}(t)(\nabla w^h_{\Delta t,i}, \nabla P_1^h\xi)dt = 0,$$
$$\int_0^T \mu_{\Delta t}(t)[(w^h_{\Delta t,i}, P_1^h\xi)^h - (F^h_{i,\Delta t}, P_1^h\xi)^h - \gamma(\nabla u^h_{\Delta t,i}, \nabla P_1^h\xi)]dt = 0.$$

Rewriting these equations as

$$
\begin{aligned}
-\int_0^T (u^h_{\Delta t,i}, P_1^h\xi)\frac{d}{dt}\bar{\mu}_{\Delta t}(t)dt + \mu(t^{N-1})(U_i^N, P_1^h\xi)^h - \mu(t^0)(u_i^0, P_1^h\xi)^h \\
+\int_0^T \mu_{\Delta t}(t)(\nabla w^h_{\Delta t,i}, \nabla P_1^h\xi)dt + \int_0^T [(u^h_{\Delta t,i}, P_1^h\xi) - (u^h_{\Delta t,i}, P_1^h\xi)^h]\frac{d}{dt}\bar{\mu}_{\Delta t}(t)dt = 0, \\
\int_0^T \mu_{\Delta t}(t)[(w^h_{\Delta t,i}, P_1^h\xi) - (F^h_{i,\Delta t}, P_1^h\xi) - \gamma(\nabla u^h_{\Delta t,i}, \nabla P_1^h\xi)]dt \\
+\int_0^T \mu_{\Delta t}(t)[(w^h_{\Delta t,i}, P_1^h\xi)^h - (w^h_{\Delta t,i}, P_1^h\xi)]dt \\
+\int_0^T \mu_{\Delta t}(t)[(F^h_{i,\Delta t}, P_1^h\xi) - (F^h_{i,\Delta t}, P_1^h\xi)^h]dt = 0,
\end{aligned}
$$

and applying (3.1.2b) and the Cauchy-Schwarz inequality we find that

$$
\begin{aligned}
&\left|\int_0^T [(u^h_{\Delta t,i}, P_1^h\xi) - (u^h_{\Delta t,i}, P_1^h\xi)^h]\frac{d}{dt}\bar{\mu}_{\Delta t}(t)dt\right| \\
&\leq Ch^2\int_0^T \|u^h_{\Delta t,i}\|_1 \, \|P_1^h\xi\|_1 dt \\
&\leq Ch^2\left(\int_0^T \|u^h_{\Delta t,i}\|_1^2\, dt\right)^{\frac{1}{2}}\left(\int_0^T \|P_1^h\xi\|_1^2\, dt\right)^{\frac{1}{2}} \\
&= Ch^2\|u^h_{\Delta t,i}\|_{L^2(0,T;H^1(\Omega))}\left(\int_0^T \|P_1^h\xi\|_1^2\, dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\left|\int_0^T \mu_{\Delta t}(t)[(w^h_{\Delta t,i}, P_1^h\xi)^h - (w^h_{\Delta t,i}, P_1^h\xi)]\, dt\right| \\
&\leq Ch^2\|w^h_{\Delta t,i}\|_{L^2(0,T;H^1(\Omega))}\left(\int_0^T \|P_1^h\xi\|_1^2\, dt\right)^{\frac{1}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| \int_0^T \mu_{\Delta t}(t)[(F^h_{\Delta t,i}, P^h_1\xi)^h - (F^h_{\Delta t,i}, P^h_1\xi)]\, dt \right| \\
&\qquad \leq Ch \int_0^T ||F^h_{i,\Delta t}||_0 ||P^h_1\xi||_1 \, \mu_{\Delta t}(t)\, dt \\
&\qquad \leq Ch \left(\int_0^T ||F^h_{i,\Delta t}||_0^2 \, dt \right)^{\frac{1}{2}} \left(\int_0^T ||P^h_1\xi||_1^2 \, dt \right)^{\frac{1}{2}} \\
&\qquad \leq Ch ||F^h_{i,\Delta t}||_{L^2(0,T;L^2(\Omega))} \left(\int_0^T ||P^h_1\xi||_1^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}
$$

Choosing μ such that $\mu(T) = 0$, $\mu(0) \neq 0$, noting (4.2.42), and observing that $P^h_1\xi \in L^2(0,T;H^1(\Omega))$ as $\Delta t, h \to 0$ and $\mu^{N-1} \to \mu(T)$, we can pass to the limit as $\Delta t, h \to 0$ to obtain

$$\int_0^T \left[-(u_i, \xi)\frac{d\mu}{dt} + \mu(t)(\nabla w_i, \nabla \xi) \right] dt - \mu(0)(u_i^0, \xi) = 0, \tag{4.2.47}$$

$$\int_0^T \mu(t)[(w_i, \xi) - (v_i, \xi) - \gamma(\nabla u_i, \nabla \xi)] dt = 0, \tag{4.2.48}$$

which implies

$$
\begin{aligned}
-(\frac{\partial u_i}{\partial t}, \xi) + (\nabla w_i, \nabla \xi) = 0 \quad & a.e. \text{ in } (0,T), \\
(w_i, \xi) - (v_i, \xi) - \gamma(\nabla u_i, \nabla \xi) = 0 \quad & a.e. \text{ in } (0,T).
\end{aligned}
$$

An integration by parts of (4.2.47) gives

$$(u_i(0) - u_i^0, \xi) = 0 \qquad \forall \xi \in H^1(\Omega),$$

and therefore $u_1(0) = u_1^0$.

It remains to prove

$$v_i = F_i(u_1, u_2). \tag{4.2.49}$$

To show (4.2.49) we note the following notation

$$\bar{u}^{h+}_{\Delta t,i} := U_i^n, \qquad \bar{u}^{h-}_{\Delta t,i} := U_i^{n-1}.$$

Now for all $\eta \in H^1(\Omega)$ consider

$$\begin{aligned}
\left|(F^h_{1,\Delta t} - v_1, \eta)\right| &= \left|(F_1(U^n_1, U^n_2) - v_1, \eta)\right| \\
&= \big|((\bar{u}^{h+}_{\Delta t,1})^3 - \bar{u}^{h-}_{\Delta t,1} + D(\bar{u}^{h+}_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1} + 2)(\bar{u}^{h-}_{\Delta t,2} + 1)^2, \eta) \\
&\quad - (u_1^3 - u_1 + 2D(u_1 + 1)(u_2 + 1)^2, \eta)\big| \\
&\le \left|((\bar{u}^{h+}_{\Delta t,1})^3 - u_1^3, \eta)\right| + \left|(u - \bar{u}^{h-}_{\Delta t,1}, \eta)\right| \\
&\quad + D\left|((\bar{u}^{h+}_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1} + 2)(\bar{u}^{h-}_{\Delta t,2} + 1)^2 - 2(u_1 + 1)(u_2 + 1)^2, \eta)\right| \\
&= I_1 + I_2 + DI_3,
\end{aligned}$$

where I_j, for $j = 1, 2, 3$, correspond to the jth-term of the right hand side. Recalling the convergence in (4.2.43), it is enough to consider I_1 and I_3.

Noting (2.1.9) and (4.2.43) we have

$$\begin{aligned}
I_1 = &\ \left|((\bar{u}^{h+}_{\Delta t,1} - u_1)((\bar{u}^{h+}_{\Delta t,1})^2 + \bar{u}^{h+}_{\Delta t,1} u_1 + u_1^2), \eta)\right| \\
\le &\ \frac{3}{2}\left|(|\bar{u}^{h+}_{\Delta t,1} - u_1|((\bar{u}^{h+}_{\Delta t,1})^2 + u_1^2), |\eta|)\right| \\
\le &\ C|\bar{u}^{h+}_{\Delta t,1} - u_1|_0\left(\|\bar{u}^{h+}_{\Delta t,1}\|_1^2 + \|\bar{u}^{h+}_{\Delta t,1}\|_1\|u_1\|_1 + \|u_1\|_1^2\right)\|\eta\|_1 \to 0 \quad \text{as} \quad \Delta t \to 0.
\end{aligned}$$

We can rewrite the equation I_3 as,

$$\begin{aligned}
I_3 &= \big|((\bar{u}^{h+}_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1} + 2)(\bar{u}^{h-}_{\Delta t,2} + 1)^2 - 2(u^h_{\Delta t,1} + 1)(\bar{u}^{h-}_{\Delta t,2} + 1)^2 \\
&\quad + 2(u^h_{\Delta t,1} + 1)(\bar{u}^{h-}_{\Delta t,2} + 1)^2 - 2(u^h_{\Delta t,1} + 1)(u^h_{\Delta t,2} + 1)^2 \\
&\quad + 2(u^h_{\Delta t,1} + 1)(u^h_{\Delta t,2} + 1)^2 - 2(u_1 + 1)(u_2 + 1)^2, \eta)\big| \\
&\le \left|((\bar{u}^{h+}_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1} + 2)(\bar{u}^{h-}_{\Delta t,2} + 1)^2 - 2(u^h_{\Delta t,1} + 1)(\bar{u}^{h-}_{\Delta t,2} + 1)^2, \eta)\right| \\
&\quad + 2\left|(u^h_{\Delta t,1} + 1)(\bar{u}^{h-}_{\Delta t,2} + 1)^2 - (u^h_{\Delta t,1} + 1)(u^h_{\Delta t,2} + 1)^2, \eta)\right| \\
&\quad + 2\left|(u^h_{\Delta t,1} + 1)(u^h_{\Delta t,2} + 1)^2 - (u_1 + 1)(u_2 + 1)^2, \eta)\right|.
\end{aligned}$$

Now we look at each terms in the right hand side. Using (2.1.9), the Hölder inequality (2.1.7) and (4.2.43) we have

$$
\begin{aligned}
\big|\big((\bar{u}^{h+}_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1}&+2)(\bar{u}^{h-}_{\Delta t,2}+1)^2 - 2(u^h_{\Delta t,1}+1)(\bar{u}^{h-}_{\Delta t,2}+1)^2,\eta\big)\big| \\
&= \big|\big((\bar{u}^{h+}_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1}+2) - 2(u^h_{\Delta t,1}+1), (\bar{u}^{h-}_{\Delta t,2}+1)^2\eta\big)\big| \\
&= |(\bar{u}^{h+}_{\Delta t,1} - u^h_{\Delta t,1} + \bar{u}^{h-}_{\Delta t,1} - u^h_{\Delta t,1}, (\bar{u}^{h-}_{\Delta t,2}+1)^2\eta)| \\
&\le |(\bar{u}^{h+}_{\Delta t,1} - u^h_{\Delta t,1})(\bar{u}^{h-}_{\Delta t,2}+1)^2,\eta)| + |(\bar{u}^{h-}_{\Delta t,1} - u^h_{\Delta t,1})(\bar{u}^{h-}_{\Delta t,2}+1)^2,\eta)| \\
&\le C|\bar{u}^{h+}_{\Delta t,1} - u^h_{\Delta t,1}|_0 \, \big(\|\bar{u}^{h-}_{\Delta t,2}\|_1^2 \, \|\eta\|_1 + |\eta|_0\big) \\
&\qquad + |\bar{u}^{h-}_{\Delta t,1} - u^h_{\Delta t,1}|_0 \, \big(\|\bar{u}^{h-}_{\Delta t,2}\|_1^2 \, \|\eta\|_1 + |\eta|_0\big) \to 0 \quad \text{as } \Delta t \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\big|\big((u^h_{\Delta t,1}+1)(\bar{u}^{h-}_{\Delta t,2}+1)^2 &- (u^h_{\Delta t,1}+1)(u^h_{\Delta t,2}+1)^2,\eta\big)\big| \\
&= \big|\big((\bar{u}^{h-}_{\Delta t,2}+1)^2 - (u^h_{\Delta t,2}+1)^2, (u^h_{\Delta t,1}+1)\eta\big)\big| \\
&= \big|\big((\bar{u}^{h-}_{\Delta t,2} - u^h_{\Delta t,2})(\bar{u}^{h-}_{\Delta t,2} + u^h_{\Delta t,2}+2)(u^h_{\Delta t,1}+1),\eta\big)\big| \\
&\le C|\bar{u}^{h-}_{\Delta t,2} - u^h_{\Delta t,2}|_0\big(\|\bar{u}^{h-}_{\Delta t,2}\|_1 \, \|u^h_{\Delta t,1}\|_1\|\eta\|_1 \\
&\qquad + \|\bar{u}^{h-}_{\Delta t,2}\|_1 \, \|\eta\|_1 + \|u^h_{\Delta t,2}\|_1 \, \|u^h_{\Delta t,1}\|_1\|\eta\|_1 \\
&\qquad + \|u^h_{\Delta t,2}\|_1 \, \|\eta\|_1 + \|u^h_{\Delta t,1}\|_1\|\eta\|_1 + |\eta|_0\big) \to 0 \quad \text{as } \Delta t \to 0.
\end{aligned}
$$

Similarly to the above analysis

$$
I_2 = |(u^h_{\Delta t,1}+1)(u^h_{\Delta t,2}+1)^2 - (u_1+1)(u_2+1)^2,\eta)| \to 0 \quad \text{as } \Delta t \to 0.
$$

These results imply that

$$
F_1(U_1^n, U_2^n) = v_1.
$$

Similarly we are able to show

$$
F_2(U_1^n, U_2^n) = v_2.
$$

Here the analysis proceeds by interchanging subscript '1' and '2' and replacing the old 'D' term with $D(\bar{u}^{h+}_{\Delta t,2} + \bar{u}^{h-}_{\Delta t,2} + 2)(\bar{u}^{h+}_{\Delta t,1} + 1)^2$. This ends the discussion of the convergence.

4.3 Scheme 2

4.3.1 Existence and Uniqueness

We consider a two-level scheme for approximating (**P**) defined as follows:

($\mathbf{P}_2^{h,\Delta t}$) Given $U_1^0, U_1^1, U_2^0, U_2^1$, find $\{U_1^{n+1}, U_2^{n+1}, W_1^n, W_2^n\} \in S^h \times S^h \times S^h \times S^h$, for $n = 1, \dots, N$, such that $\forall \eta \in S^h$

$$\left(\frac{U_1^{n+1} - U_1^{n-1}}{2\Delta t}, \eta\right)^h = -\,(\nabla W_1^n, \nabla\eta), \tag{4.3.1a}$$

$$(W_1^n, \eta)^h = (F_1(U_1^{n+1}, U_2^{n+1}),\eta)^h + \gamma\left(\nabla\Big(\frac{U_1^{n+1} + U_1^{n-1}}{2}\Big), \nabla\eta\right), \tag{4.3.1b}$$

$$U_1^0 = P^h u_1^0, \qquad U_1^1 = P^h u_1^1, \tag{4.3.1c}$$

and

$$\left(\frac{U_2^{n+1} - U_2^{n-1}}{2\Delta t}, \eta\right)^h = -\,(\nabla W_2^n, \nabla\eta), \tag{4.3.1d}$$

$$(W_2^n, \eta)^h = (F_2(U_1^{n+1}, U_2^{n+1}),\eta)^h + \gamma\left(\nabla\Big(\frac{U_2^{n+1} + U_2^{n-1}}{2}\Big), \nabla\eta\right), \tag{4.3.1e}$$

$$U_2^0 = P^h u_2^0, \qquad U_2^1 = P^h u_2^1, \tag{4.3.1f}$$

where

$$F_1(U_1^{n+1}, U_2^{n+1}) = (U_1^n)^2\Big(\frac{U_1^{n+1} + U_1^{n-1}}{2}\Big) - U_1^n + D(U_1^{n+1} + U_1^{n-1} + 2)(U_2^n + 1)^2, \tag{4.3.1g}$$

$$F_2(U_1^{n+1}, U_2^{n+1}) = (U_2^n)^2\Big(\frac{U_2^{n+1} + U_2^{n-1}}{2}\Big) - U_2^n + D(U_2^{n+1} + U_2^{n-1} + 2)(U_1^n + 1)^2. \tag{4.3.1h}$$

As we shall show, this is a linear scheme which generates a stable unique sequence. The treatment of cubic terms in Scheme 2 is based on work by Matsuo and Furihata [21].

Similarly to (4.2.3) and (4.2.4) we have

$$⨍ W_i^n = \frac{1}{|\Omega|}(W_i^n,1)^h = \frac{1}{|\Omega|}(F_i(U_1^{n+1},U_2^{n+1}),1)^h \quad \text{for } i=1,2, \tag{4.3.2}$$

and

$$W_i^n = -\widehat{\mathcal{G}}^h[\frac{U_i^{n+1}-U_i^{n-1}}{2\Delta t}] + ⨍ W_i^n. \tag{4.3.3}$$

Noting (4.3.3), (4.3.2), and (4.2.5) we can restate the problem $(\mathbf{P}_2^{h,\Delta t})$ as: Given $U_1^0, U_1^1, U_2^0, U_2^1$, find $\{U_1^{n+1}, U_2^{n+1}\} \in S^h \times S^h$, for $n=1,\ldots,N$ such that $\forall \eta \in S^h$

$$(\widehat{\mathcal{G}}^h[\frac{U_1^{n+1}-U_1^{n-1}}{2\Delta t}],\eta)^h + \gamma(\nabla(\frac{U_1^{n+1}+U_1^{n-1}}{2}),\nabla\eta)$$
$$+ (F_1(U_1^{n+1},U_2^{n+1}),(I-⨍\,)\eta)^h = 0, \tag{4.3.4a}$$
$$(\widehat{\mathcal{G}}^h[\frac{U_2^{n+1}-U_2^{n-1}}{2\Delta t}],\eta)^h + \gamma(\nabla(\frac{U_2^{n+1}+U_2^{n-1}}{2}),\nabla\eta)$$
$$+ (F_2(U_1^{n+1},U_2^{n+1}),(I-⨍\,)\eta)^h = 0. \tag{4.3.4b}$$

Theorem 4.3.1 Let $\Delta t > 0$ and the assumptions **(A)** hold. Given $u_1^0, u_1^1, u_2^0, u_2^1 \in S^h$ such that $\|u_1^0\|_1 + \|u_1^1\|_1 + \|u_2^0\|_1 + \|u_2^1\|_1 \le C$. Then for all $h>0$ there exists a unique solution $\{U_1, U_2, W_1, W_2\}$ to $(\mathbf{P}_2^{h,\Delta t})$ such that

$$\max_{m=1,\ldots,N}\Big\{2\Delta t\sum_{n=1}^{m}(|W_1^n|_1^2 + |W_1^n|_1^2) + \frac{\gamma}{2}(|U_1^{m+1}|_1^2 + |U_2^{m+1}|_1^2)$$
$$+ D\big[\big((U_1^{m+1}+1)^2(U_2^m+1)^2,1\big)^h + \big((U_1^m+1)^2(U_2^{m+1}+1)^2,1\big)^h\big]$$
$$+\frac{1}{2}\big(|U_1^mU_1^{m+1}|_h^2 + |U_2^mU_2^{m+1}|_h^2\big)\Big\} < C, \tag{4.3.5}$$

$$2\Delta t\sum_{n=1}^{N}\|W_1^n\|_1^2 + 2\Delta t\sum_{n=1}^{N}\|W_2^n\|_1^2 \le C, \tag{4.3.6}$$

$$|F_1(U_1^{n+1},U_2^{n+1})|_h^2 + |F_2(U_1^{n+1},U_2^{n+1})|_h^2 \le C \quad \text{for } d=1,2. \tag{4.3.7}$$

Proof. Let

$$K_1^h = \{\eta_1 \in S^h : (\eta_1, 1)^h = m_1 := \fint u_1^0\}, \tag{4.3.8a}$$

$$K_2^h = \{\eta_2 \in S^h : (\eta_2, 1)^h = m_2 := \fint u_2^0\}. \tag{4.3.8b}$$

Consider the uncoupled variational problems

$$\min_{\eta_1 \in K_1^h} \quad \mathcal{E}_1^h(\eta_1), \qquad \min_{\eta_2 \in K_2^h} \quad \mathcal{E}_2^h(\eta_2),$$

where

$$\begin{aligned}
\mathcal{E}_1^h(\eta_1) =& \frac{1}{2}|\widehat{\mathcal{G}}^h[\frac{\eta_1 - U_1^{n-1}}{2\Delta t}]|_1^2 + \frac{\gamma}{4}|\eta_1|_1^2 + \frac{\gamma}{2}(\nabla U_1^{n-1}, \nabla\eta_1) + ((U_1^n)^2, \frac{1}{4}\eta_1^2 + \frac{1}{2}U_1^{n-1}\eta_1)^h \\
&- (U_1^n, \eta_1)^h + D((U_2^n+1)^2, \frac{1}{2}\eta_1^2 + (U_1^{n-1}+2)\eta_1)^h, \\
\mathcal{E}_2^h(\eta_2) =& \frac{1}{2}|\widehat{\mathcal{G}}^h[\frac{\eta_2 - U_2^{n-1}}{2\Delta t}]|_1^2 + \frac{\gamma}{4}|\eta_2|_1^2 + \frac{\gamma}{2}(\nabla U_2^{n-1}, \nabla\eta_2) + ((U_2^n)^2, \frac{1}{4}\eta_2^2 + \frac{1}{2}U_2^{n-1}\eta_2)^h \\
&- (U_2^n, \eta_2)^h + D((U_1^n+1)^2, \frac{1}{2}\eta_2^2 + (U_2^{n-1}+2)\eta_2)^h.
\end{aligned}$$

Since $|\widehat{\mathcal{G}}^h[\frac{\eta_i - U_i^{n-1}}{2\Delta t}]|_1^2 \geq 0$ for $i = 1, 2$, we have

$$\begin{aligned}
\mathcal{E}_1^h(\eta_1) \geq& \frac{\gamma}{4}|\eta_1|_1^2 + \frac{\gamma}{2}(\nabla U_1^{n-1}, \nabla\eta_1) + \frac{1}{4}((U_1^n)^2, \eta_1^2)^h + \frac{1}{2}((U_1^n)^2, U_1^{n-1}\eta_1)^h - (U_1^n, \eta_1)^h \\
&+ \frac{D}{2}((U_2^n+1)^2, \eta_1^2)^h + D((U_2^n+1)^2, (U_1^{n-1}+2)\eta_1)^h.
\end{aligned} \tag{4.3.9}$$

By noting the Young inequality (2.1.4), for $p = q = 2$ we have

$$\begin{aligned}
(\nabla U_1^{n-1}, \nabla\eta_1) &\leq 2|U_1^{n-1}|_1^2 + \frac{1}{8}|\eta_1|_1^2, \\
(U_1^n, \eta_1)^h &\leq \frac{1}{4}((U_1^n)^2, (\eta_1)^2)^h + |\Omega|, \\
((U_2^n+1)^2, (U_1^{n-1}+2)\eta_1)^h &\leq \frac{1}{2}((U_2^n+1)^2, (\eta_1)^2)^h + \frac{1}{2}((U_2^n+1)^2, (U_1^{n-1}+2)^2)^h.
\end{aligned}$$

Using the Young inequality (2.1.4), for $p = q = 2$, the Poincaré inequality (3.1.3) and setting $\epsilon = 4C/\gamma$, we have

$$\frac{1}{2}((U_1^n)^2, U_1^{n-1}\eta_1)^h \leq \frac{2C}{\gamma}((U_1^n)^4, (U_1^{n-1})^2)^h + \frac{\gamma}{16}|\eta_1|_1^2 + \frac{\gamma}{16}|(\eta_1, 1)^h|^2.$$

Hence

$$\begin{aligned}
\frac{\gamma}{2}|(\nabla U_1^{n-1}, \nabla\eta_1)| &\geq -\gamma|U_1^{n-1}|_1^2 - \frac{\gamma}{18}|\eta_1|_1^2, &(4.3.10)\\
|(U_1^n, \eta_1)^h| &\geq -\frac{1}{4}((U_1^n)^2, (\eta_1)^2)^h - |\Omega|, &(4.3.11)\\
|D((U_2^n+1)^2, (U_1^{n-1}+2)\eta_1)^h| &\geq -\frac{D}{2}((U_2^n+1)^2, (\eta_1)^2)^h\\
&\quad - \frac{D}{2}((U_2^n+1)^2, (U_1^{n-1}+2)^2)^h, &(4.3.12)\\
|\frac{1}{2}((U_1^n)^2, U_1^{n-1}\eta_1)^h| &\geq -\frac{2C}{\gamma}((U_1^n)^4, (U_1^{n-1})^2)^h\\
&\quad - \frac{\gamma}{16}|\eta_1|_1^2 - \frac{\gamma}{16}|(\eta_1, 1)^h|^2. &(4.3.13)
\end{aligned}$$

Noting (4.3.10–4.3.13) we can rewrite (4.3.9) as

$$\begin{aligned}
\mathcal{E}_1^h(\eta_1) \geq& \frac{\gamma}{8}|\eta_1|_1^2 - \gamma|U_1^{n-1}|_1^2 - \frac{2C}{\gamma}\big((U_1^n)^4, (U_1^{n-1})^2\big)^h\\
&- \frac{D}{2}\big((U_2^n+1)^2, (U_1^{n-1}+2)^2\big)^h - \frac{\gamma}{16}|(\eta_1, 1)^h|^2 - |\Omega|. \qquad (4.3.14)
\end{aligned}$$

Similarly we are able to show

$$\begin{aligned}
\mathcal{E}_2^h(\eta_2) \geq& \frac{\gamma}{8}|\eta_2|_1^2 - \gamma|U_2^{n-1}|_1^2 - \frac{2C}{\gamma}\big((U_2^n)^4, (U_2^{n-1})^2\big)^h\\
&- \frac{D}{2}\big((U_1^n+1)^2, (U_2^{n-1}+2)^2\big)^h - \frac{\gamma}{16}|(\eta_2, 1)^h|^2 - |\Omega|. \qquad (4.3.15)
\end{aligned}$$

Thus $\mathcal{E}_i^h$ are bounded below and it follows that there exist solutions to the coupled variational problems. Such minimisers, which are critical points of $\mathcal{E}_i^h$, satisfy the Euler–Lagrange equation of $\mathcal{E}_i^h$ given by (4.3.4a) and (4.3.4b) respectively, while $\{W_1, W_2\}$ exist from (4.3.3) and (4.3.2).

To show the uniqueness, let $U_1^{n+1,1}, U_2^{n+1,1}$ and $U_1^{n+1,2}, U_2^{n+1,2}$ be two solutions of

$(\mathbf{P}_2^{h,\Delta t})$ and set

$$Z_1 = U_1^{n+1,1} - U_1^{n+1,2}, \qquad Z_2 = U_2^{n+1,1} - U_2^{n+1,2}. \tag{4.3.16}$$

From (4.3.4a) we have, for $i = 1, 2$,

$$\begin{aligned}(\widehat{\mathcal{G}}^h[\frac{U_1^{n+1,i} - U_1^{n-1,i}}{2\Delta t}], \eta)^h + \gamma(\nabla(\frac{U_1^{n+1,i} + U_1^{n-1,i}}{2}), \nabla\eta)& \\ + (F_1(U_1^{n+1,i}, U_2^{n+1,i}), \eta - \textstyle\fint \eta)^h = 0.& \qquad (4.3.17)\end{aligned}$$

Subtracting these equations and noting (4.3.16) yields

$$\begin{aligned}&\frac{1}{2\Delta t}(\widehat{\mathcal{G}}^h Z_1, \eta)^h + \frac{\gamma}{2}(\nabla Z_1, \nabla\eta) \\ &\quad + (F_1(U_1^{n+1,1}, U_2^{n+1,1}) - F_1(U_1^{n+1,2}, U_2^{n+1,2}), \eta - \textstyle\fint \eta)^h = 0. \qquad (4.3.18)\end{aligned}$$

Taking $\eta = Z_1$ in (4.3.18) and noting (4.1.2) and (4.3.16), we have

$$\frac{1}{2\Delta t}|\widehat{\mathcal{G}}^h Z_1|_1^2 + \frac{\gamma}{2}|Z_1|_1^2 + (F_1(U_1^{n+1,1}, U_2^{n+1,1}) - F_1(U_1^{n+1,2}, U_2^{n+1,2}), Z_1)^h = 0.$$

Note that

$$\begin{aligned}&F_1(U_1^{n+1,1}, U_2^{n+1,1}) - F_1(U_1^{n+1,2}, U_2^{n+1,2}) \\ &\quad = (U_1^{n,1})^2(\frac{U_1^{n+1,1} + U_1^{n-1,1}}{2}) - U_1^{n,1} + D(U_1^{n+1,1} + U_1^{n-1,1} + 2)(U_2^{n,1} + 1)^2 \\ &\qquad - (U_1^{n,2})^2(\frac{U_1^{n+1,2} + U_1^{n-1,2}}{2}) + U_1^{n,2} - D(U_1^{n+1,2} + U_1^{n-1,2} + 2)(U_2^{n,2} + 1)^2 \\ &\quad = ((U_1^n)^2 + D(U_2^n + 1)^2)Z_1. \qquad (4.3.19)\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2\Delta t}|\widehat{\mathcal{G}}^h Z_1|_1^2 + \frac{\gamma}{2}|Z_1|_1^2 + (((U_1^n)^2 + D(U_2^n + 1)^2)Z_1, Z_1)^h = 0,& \\ \frac{1}{2\Delta t}|\widehat{\mathcal{G}}^h Z_1|_1^2 + \frac{\gamma}{2}|Z_1|_1^2 + ((U_1^n)^2 + D(U_2^n + 1)^2)|Z_1|_h^2 = 0.&\end{aligned}$$

It follows from the Poincaré inequality (3.1.3) that

$$|Z_1|_h^2 = 0, \quad \text{i.e.} \quad Z_1 \equiv 0.$$

In the same way, we are able to show that

$$Z_2 \equiv 0.$$

Thus we have shown the uniqueness.

Now we deduce the stability bound (4.3.5). Choosing $\eta = W_i^n$, for $i = 1, 2$, in (4.3.1a, 4.3.1d) and $\eta = U_i^{n+1} - U_i^{n-1}$ in (4.3.1b, 4.3.1e), we obtain

$$(U_i^{n+1} - U_i^{n-1}, W_i^n)^h = -2\Delta t|W_i^n|_1^2, \tag{4.3.20}$$

$$\begin{aligned}(W_i^n, U_i^{n+1} - U_i^{n-1})^h &= (F_i(U_1^{n+1}, U_2^{n+1}), U_i^{n+1} - U_i^{n-1})^h \\ &+ \gamma(\nabla(\frac{U_i^{n+1} + U_i^{n-1}}{2}), \nabla(U_i^{n+1} - U_i^{n-1})).\end{aligned} \tag{4.3.21}$$

Adding (4.3.21) to (4.3.20) yields

$$2\Delta t|W_i^n|_1^2 + \frac{\gamma}{2}(|U_i^{n+1}|_1^2 - |U_i^{n-1}|_1^2) = (F_i(U_1^{n+1}, U_2^{n+1}), U_i^{n-1} - U_i^{n+1})^h. \tag{4.3.22}$$

Note that, for $i, j = 1, 2,\ i \neq j$, we obtain

$$\begin{aligned}((U_i^n)^2, (U_i^{n-1})^2 - (U_i^{n+1})^2)^h &= ((U_i^{n-1})^2, (U_i^n))^h - ((U_i^n)^2, (U_i^{n+1})^2)^h \\ &= |U_i^{n-1}U_i^n|_h^2 - |U_i^nU_i^{n+1}|_h^2,\end{aligned} \tag{4.3.23}$$

$$(U_i^n, U_i^{n+1} - U_i^{n-1})^h = (U_i^n, U_i^{n+1})^h - (U_i^{n-1}, U_i^n)^h, \tag{4.3.24}$$

and

$$\begin{aligned}&((U_i^{n+1} + U_i^{n-1} + 2)(U_j^n + 1)^2, U_i^{n-1} - U_i^{n+1})^h \\ &= ((U_i^{n-1} + 1)^2, (U_j^n + 1)^2)^h - ((U_i^{n+1} + 1)^2, (U_j^n + 1)^2)^h.\end{aligned} \tag{4.3.25}$$

Adding (4.3.22) for $i = 1, 2$, and noting (4.3.23–4.3.25) we have

$$
\begin{aligned}
&2\Delta t(|W_1^n|_1^2 + |W_2^n|_1^2) + \frac{\gamma}{2}\big(|U_1^{n+1}|_1^2 - |U_1^{n-1}|_1^2 + |U_2^{n+1}|_1^2 - |U_2^{n-1}|_1^2\big)\\
&\quad + \frac{1}{2}\big(|U_1^n U_1^{n+1}|_h^2 - |U_1^{n-1}U_1^n|_h^2 + |U_2^n U_2^{n+1}|_h^2 - |U_2^{n-1}U_2^n|_h^2\big)\\
&\quad + \big(U_1^n, U_1^{n+1}\big)^h - \big(U_1^{n-1}, U_1^n\big)^h + \big(U_2^n, U_2^{n+1}\big)^h - \big(U_2^{n-1}, U_2^n\big)^h\\
&\quad + D\big[\big((U_1^{n+1}+1)^2, (U_2^n+1)^2\big)^h - \big((U_1^{n-1}+1)^2, (U_2^n+1)^2\big)^h\\
&\quad + \big((U_2^{n+1}+1)^2, (U_1^n+1)^2\big)^h - \big((U_2^{n-1}+1)^2, (U_1^n+1)^2\big)^h\big] = 0. \qquad (4.3.26)
\end{aligned}
$$

Summing (4.3.26) for $n = 1, \ldots, m$, $m \leq N$ and rearranging the terms we obtain

$$
\begin{aligned}
&\max_{m=1,\ldots,N} 2\Delta t\Big\{\sum_{n=1}^{m}(|W_1^n|_1^2 + |W_1^n|_1^2) + \frac{\gamma}{2}\big(|U_1^{m+1}|_1^2 + |U_2^{m+1}|_1^2\big)\\
&\quad + D\big[\big((U_1^{m+1}+1)^2(U_2^m+1)^2, 1\big)^h + \big((U_1^m+1)^2(U_2^{m+1}+1)^2, 1\big)^h\big]\\
&\quad + \frac{1}{2}\big(|U_1^m U_1^{m+1}|_h^2 + |U_2^m U_2^{m+1}|_h^2\big)\Big\}\\
&\qquad \leq \frac{\gamma}{2}\big(|U_1^0|_1^2 + |U_1^1|_1^2 + |U_2^0|_1^2 - |U_2^1|_1^2\big) + \frac{1}{2}\big(|U_1^0 U_1^1|_h^2 + |U_2^0 U_2^1|_h^2\big)\\
&\qquad\quad + D\big[\big((U_1^0+1)^2(U_2^1+1)^2, 1\big)^h + \big((U_1^1+1)^2(U_2^0+1)^2, 1\big)^h\big] < C,
\end{aligned}
$$

where to obtain the last inequality we have noted the Young inequality (2.1.4), (3.1.40), (4.3.1c), (4.3.1f), (3.1.14) and the condition on u_i^k for $i = 1, 2$, $k = 0, 1$.

The proofs of (4.3.6) and (4.3.7) are analogues to those given for (4.2.8) and (4.2.9) respectively. This ends the proof. □

4.4 Error bound

In this section we derive the error estimate between the semidiscrete approximations u_i^h defined by (3.2.1a–f) and their fully discrete approximation U_i^n defined by (4.2.1a–f), Scheme 1. We introduce the following variables

$$
U_i(t) := \frac{t - t^{n-1}}{\Delta t} U_i^n + \frac{t^n - t}{\Delta t} U_i^{n-1}, \quad t \in [t^{n-1}, t^n] \;\; n \geq 1, \qquad (4.4.1)
$$

and

$$\widehat{U}_i(t) := U_i^n \qquad t \in (t^{n-1}, t^n) \;\; n \geq 1. \tag{4.4.2}$$

Using this notation we can restate the problem $(\mathbf{P}^{h,\Delta t})$ as:
Find $\{U_1, U_2\} \in H^1(0,T;S^h) \times H^1(0,T;S^h)$, such that $U_i(0) \equiv P^h u_i^0$ and for *a.e.* $t \in (0,T)$, $(U_i(t),1) = (u_i^0,1)$ and

$$(\widehat{\mathcal{G}}^h \frac{\partial U_1}{\partial t}, \eta)^h + \gamma(\nabla \widehat{U}_1, \nabla \eta) + (F_1(\widehat{U}_1, \widehat{U}_2), (I - ⨍\,)\eta)^h = 0, \tag{4.4.3}$$
$$(\widehat{\mathcal{G}}^h \frac{\partial U_2}{\partial t}, \eta)^h + \gamma(\nabla \widehat{U}_2, \nabla \eta) + (F_2(\widehat{U}_1, \widehat{U}_2), (I - ⨍\,)\eta)^h = 0, \tag{4.4.4}$$

for all $\eta \in S^h$.

Theorem 4.4.1 Let the assumptions of Theorem (4.2.1) hold. Then we have that for $i = 1, 2$,

$$\|u_i^h - U_i\|_{L^\infty(0,T;H^1(\Omega)')} + \|u_i^h - \widehat{U}_i\|_{L^2(0,T;H^1(\Omega))} \leq C\Big[\Delta t + \frac{h^4}{\Delta t}\Big]. \tag{4.4.5}$$

Proof. Let $E_i := u_i^h - U_i$, $\widehat{E}_i := u_i^h - \widehat{U}_i$, $\widehat{E}_i^- := u_i^h - U_i^{n-1} \in V^h$ for *a.e.* $t \in (0,T)$. Subtracting (4.4.3) and (4.4.4) from (3.2.5a) and (3.2.5b) respectively we have

$$(\mathcal{G}^h \frac{\partial u_i^h}{\partial t}, \eta) + \gamma(\nabla \widehat{E}_i, \nabla \eta)$$
$$= (\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \eta)^h + \big(F_i(\widehat{U}_1, \widehat{U}_2) - \phi(u_i^h) + 2D\Psi_i(u_1^h, u_2^h), (I - ⨍\,)\eta\big)^h.$$

Setting $\eta = \widehat{E}_i$ and recalling that mass is conserved, i.e. $⨍\, \widehat{E}_i = 0$, we obtain

$$(\mathcal{G}^h \frac{\partial u_i^h}{\partial t}, \widehat{E}_i) + \gamma(\nabla \widehat{E}_i, \nabla \widehat{E}_i)$$
$$= (\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)^h + \big(F_i(\widehat{U}_1, \widehat{U}_2) - \phi(u_i^h) + 2D\Psi_i(u_1^h, u_2^h), \widehat{E}_i\big)^h.$$

Subtracting $(\mathcal{G}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)$, from both sides, simplifying and rearranging the terms we have

$$(\mathcal{G}^h \frac{\partial}{\partial t} E_i, u_i^h) + \gamma |\widehat{E}_i|_1^2 = (\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)^h - (\mathcal{G}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i) + (\mathcal{G}^h \frac{\partial}{\partial t} E_i, \widehat{U}_i)$$
$$+ \big(F_i(\widehat{U}_1, \widehat{U}_2) - \phi(u_i^h) + 2D\Psi_i(u_1^h, u_2^h), \widehat{E}_i\big)^h.$$

Subtracting $(\mathcal{G}^h \frac{\partial}{\partial t} E_i, U_i)$ from both sides, simplifying, noting (3.2.47) and $E_i - \widehat{E}_i = \widehat{U}_i - U_i$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_i\|_{-h}^2 + \gamma |\widehat{E}_i|_1^2 = (\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)^h - (\mathcal{G}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i) + (\mathcal{G}^h \frac{\partial}{\partial t} E_i, E_i - \widehat{E}_i)$$
$$+ \big(F_i(\widehat{U}_1, \widehat{U}_2) - \phi(u_i^h) + 2D\Psi_i(u_1^h, u_2^h), \widehat{E}_i\big)^h.$$

Subtracting and adding $(\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)$ to the right hand side equations above and simplifying we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E_i\|_{-h}^2 + \gamma |\widehat{E}_i|_1^2 = {} & \big[(\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)^h - (\widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t}, \widehat{E}_i)\big] \\ & + ((\widehat{\mathcal{G}}^h - \mathcal{G}^h)\frac{\partial U_i}{\partial t}, \widehat{E}_i) + (\mathcal{G}^h \frac{\partial E_i}{\partial t}, E_i - \widehat{E}_i) \\ & + \big(F_i(\widehat{U}_1, \widehat{U}_2) - \phi(u_i^h) + 2D\Psi_i(u_1^h, u_2^h), \widehat{E}_i\big)^h.\end{aligned}$$

Summing both sides for $i = 1, 2$, and rearranging the terms we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|E_1\|_{-h}^2 + \|E_2\|_{-h}^2\big) + \gamma\big(|\widehat{E}_1|_1^2 + |\widehat{E}_2|_1^2\big) \\ &\quad = \big[(\widehat{\mathcal{G}}^h \frac{\partial U_1}{\partial t}, \widehat{E}_1)^h - (\widehat{\mathcal{G}}^h \frac{\partial U_1}{\partial t}, \widehat{E}_1)\big] \\ &\quad\quad + ((\widehat{\mathcal{G}}^h - \mathcal{G}^h)\frac{\partial U_1}{\partial t}, \widehat{E}_1) + (\mathcal{G}^h \frac{\partial E_1}{\partial t}, E_1 - \widehat{E}_1) \\ &\quad\quad + \big[(\widehat{\mathcal{G}}^h \frac{\partial U_2}{\partial t}, \widehat{E}_2)^h - (\widehat{\mathcal{G}}^h \frac{\partial U_2}{\partial t}, \widehat{E}_2)\big] \\ &\quad\quad + ((\widehat{\mathcal{G}}^h - \mathcal{G}^h)\frac{\partial U_2}{\partial t}, \widehat{E}_2) + (\mathcal{G}^h \frac{\partial E_2}{\partial t}, E_2 - \widehat{E}_2) \\ &\quad\quad + \big(F_1(\widehat{U}_1, \widehat{U}_2) - \phi(u_1^h) + 2D\Psi_1(u_1^h, u_2^h), \widehat{E}_1\big)^h \\ &\quad\quad + \big(F_2(\widehat{U}_1, \widehat{U}_2) - \phi(u_2^h) + 2D\Psi_2(u_1^h, u_2^h), \widehat{E}_2\big)^h, \\ &\quad = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned} \tag{4.4.6}$$

where I_j corresponds to the jth-line of the terms on the right hand side.

Now we bound each I_j in turn. Using (3.1.2b), the Poincaré inequality (2.1.2), and the Young inequality (2.1.4) we have, for $j = 1, 3$,

$$I_j \le Ch^2 \left\| \widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t} \right\|_1 \|\widehat{E}_i\|_1 \le Ch^2 \left\| \widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t} \right\|_1 |\widehat{E}_i|_1 \le Ch^4 \left\| \widehat{\mathcal{G}}^h \frac{\partial U_i}{\partial t} \right\|_1^2 + \frac{1}{\epsilon} |\widehat{E}_i|_1^2. \quad (4.4.7)$$

Notice that for $t \in (t^{n-1}, t^n)$, using (4.4.1) and (4.4.2) we have

$$\begin{aligned}
\widehat{E}_i^- - E_i &= U_i - U_i^{n-1} \\
&= \frac{t - t^{n-1}}{\Delta t} U_i^n + \frac{t^n - t}{\Delta t} U_i^{n-1} - \frac{t^n - t^{n-1}}{\Delta t} U_i^{n-1} \\
&= \frac{t - t^{n-1}}{\Delta t} U_i^n - \frac{t - t^{n-1}}{\Delta t} U_i^{n-1} \\
&= \frac{t - t^{n-1}}{\Delta t} (U_i^n - U_i^{n-1}) \\
&= (t - t^{n-1}) \frac{\partial U_i^n}{\partial t}, \qquad (4.4.8)
\end{aligned}$$

and similarly

$$\widehat{E}_i - E_i = U_i - U_i^n = (t - t^n) \frac{\partial U_i^n}{\partial t}. \qquad (4.4.9)$$

Thus using the Cauchy-Schwarz inequality, (3.1.5), the Poincaré inequality (2.1.2), the Young inequality (2.1.4), (3.1.5), and (4.1.5) we have, for $j = 2, 4$,

$$\begin{aligned}
I_j &\le \left| (\widehat{\mathcal{G}}^h - \mathcal{G}^h) \frac{\partial U_i}{\partial t} \right|_0 |\widehat{E}_i|_0 + \left| (\mathcal{G}^h \frac{\partial E_i}{\partial t}, E_i - \widehat{E}_i) \right| \\
&= \left| (\widehat{\mathcal{G}}^h - \mathcal{G}^h) \frac{\partial U_i}{\partial t} \right|_0 |\widehat{E}_i|_1 + \left| (\nabla \mathcal{G}^h \frac{\partial E_i}{\partial t}, \nabla \mathcal{G}^h (E_i - \widehat{E}_i)) \right| \\
&\le C \left\| (\widehat{\mathcal{G}}^h - \mathcal{G}^h) \frac{\partial U_i}{\partial t} \right\|_1^2 + \frac{1}{\epsilon} |\widehat{E}_i|_1^2 + C \left| \mathcal{G}^h \frac{\partial E_i}{\partial t} \right|_1 |\mathcal{G}^h (E_i - \widehat{E}_i)|_1 \\
&\le Ch^4 \left\| \frac{\partial U_i}{\partial t} \right\|_1^2 + \frac{1}{\epsilon} |\widehat{E}_i|_1^2 + C \left\| \frac{\partial E_i}{\partial t} \right\|_{-h} \|U_i - \widehat{U}_i\|_{-h} \\
&\le Ch^4 \left| \frac{\partial U_i}{\partial t} \right|_1^2 + \frac{1}{\epsilon} |\widehat{E}_i|_1^2 + C \left\| \frac{\partial E_i}{\partial t} \right\|_{-h} \|U_i - \widehat{U}_i\|_{-h}. \qquad (4.4.10)
\end{aligned}$$

Now we bound I_5 and I_6. Noting

$$((U_i^n)^3 - (u_i^h)^3, \widehat{E}_i)^h \le 0,$$

expanding F_i and rearranging the terms we obtain

$$\begin{aligned} I_5 &\le \left(u_1^h - U_1^{n-1}, \widehat{E}_1\right)^h + D\left((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + 1)^2 - 2(u_1^h + 1)(u_2^h + 1)^2, \widehat{E}_1\right)^h \\ &= I_{5,1} + DI_{5,2}, \end{aligned} \tag{4.4.11}$$

and

$$\begin{aligned} I_6 &\le \left(u_2^h - U_2^{n-1}, \widehat{E}_2\right)^h + D\left((U_2^n + U_2^{n-1} + 2)(U_1^n + 1)^2 - 2(u_2^h + 1)(u_1^h + 1)^2), \widehat{E}_2\right)^h \\ &= I_{6,1} + DI_{6,2}. \end{aligned} \tag{4.4.12}$$

Also, noting the Cauchy-Schwarz inequality, (3.1.2a), the Poincaré inequality (2.1.2), the Young inequality (2.1.4), (4.4.8), (3.1.6), and reapplying the Poincaré inequality (2.1.2) we obtain, for $i, j = 1, 2$,

$$\begin{aligned} (\widehat{E}_i^-, \widehat{E}_j)^h &\le |\widehat{E}_i^-|_h \, |\widehat{E}_j|_h \\ &\le C|\widehat{E}_i^- - E_i + E_i|_0 \, |\widehat{E}_j|_0 \\ &\le C\left(|\widehat{E}_i^- - E_i|_0 + |E_i|_0\right) |\widehat{E}_j|_1 \\ &\le C|\widehat{E}_i^- - E_i|_0^2 + C|E_i|_0^2 + \frac{1}{\epsilon}|\widehat{E}_j|_1^2 \\ &= C\left|(t - t^{n-1})\frac{\partial U_i^n}{\partial t}\right|_0^2 + C|E_i|_0^2 + \frac{1}{\epsilon}|\widehat{E}_j|_1^2 \\ &\le C(\Delta t)^2\left|\frac{\partial U_i^n}{\partial t}\right|_1^2 + C\|E_i\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_j|_1^2. \end{aligned} \tag{4.4.13}$$

Using the same method we obtain

$$\left(\widehat{E}_i, \widehat{E}_j\right) \le C(\Delta t)^2\left|\frac{\partial U_i^n}{\partial t}\right|_1^2 + C\|E_i\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_j|_1^2, \tag{4.4.14}$$

where we have used (4.4.9) instead of (4.4.8) in our derivation. Using (4.4.13) we have

$$I_{5,1} \le C(\Delta t)^2\left|\frac{\partial U_1^n}{\partial t}\right|_1^2 + C\|E_1\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_1|_1^2, \tag{4.4.15}$$

and

$$I_{6,1} \leq C(\Delta t)^2 \left|\frac{\partial U_2^n}{\partial t}\right|_1^2 + C\|E_2\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_2|_1^2. \tag{4.4.16}$$

Now by subtracting and adding $(U_1^n + U_1^{n-1} + 2)(u_2^h + 1)^2$ to $I_{5,2}$ and rearranging the terms we can express $I_{5,2}$ as

$$\begin{aligned}
I_{5,2} &= \big((U_1^n + U_1^{n-1} + 2)((U_2^{n-1} + 1)^2 - (u_2^h + 1)^2), \widehat{E}_1\big)^h \\
&\quad + \big((U_1^n + U_1^{n-1} - 2u_1^h)(u_2^h + 1)^2, \widehat{E}_1\big)^h \\
&= \big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)(U_2^{n-1} - u_2^h), \widehat{E}_1\big)^h \\
&\quad - \big((\widehat{E}_1 + \widehat{E}_1^-)(u_2^h + 1)^2, \widehat{E}_1\big)^h \\
&\leq -\big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-, \widehat{E}_1\big)^h - \big((u_2^h + 1)^2\widehat{E}_1^-, \widehat{E}_1\big)^h \\
&\leq \big|\big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-, \widehat{E}_1\big) \\
&\quad - \big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-, \widehat{E}_1\big)^h\big| \\
&\quad + \big|\big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-, \widehat{E}_1\big)\big| \\
&\quad + \big|\big((u_2^h + 1)^2\widehat{E}_1^-, \widehat{E}_1\big) - \big((u_2^h + 1)^2\widehat{E}_1^-, \widehat{E}_1\big)^h\big| + \big|\big((u_2^h + 1)^2\widehat{E}_1^-, \widehat{E}_1\big)\big| \\
&= \left|\int_\Omega (I - \pi^h)\big[(U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-\widehat{E}_1\big]\,dx\right| \\
&\quad + \big|\big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-, \widehat{E}_1\big)\big| \\
&\quad + \left|\int_\Omega (I - \pi^h)\big[(u_2^h + 1)^2\widehat{E}_1^-\widehat{E}_1\big]\,dx\right| + \big|\big((u_2^h + 1)^2\widehat{E}_1^-, \widehat{E}_1\big)\big| \\
&= I_{5,2,1} + I_{5,2,2} + I_{5,2,3} + I_{5,2,4},
\end{aligned} \tag{4.4.17}$$

where $I_{5,2,k}$ for $k = 1, \ldots, 4$ corresponds to kth-term in the right hand side.

Similarly we obtain

$$
\begin{aligned}
I_{6,2} &= \big((U_2^n + U_2^{n-1} + 2)((U_1^n + 1)^2 - (u_1^h + 1)^2), \widehat{E}_2\big)^h \\
&\quad + \big((U_2^n + U_2^{n-1} - 2u_2^h)(u_1^h + 1)^2, \widehat{E}_2\big)^h \\
&= \big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)(U_1^n - u_1^h), \widehat{E}_2\big)^h - \big((\widehat{E}_2 + \widehat{E}_2^-)(u_1^h + 1)^2, \widehat{E}_2\big)^h \\
&\leq -\big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)\widehat{E}_1, \widehat{E}_2\big)^h - \big((u_1^h + 1)^2 \widehat{E}_2^-, \widehat{E}_2\big)^h \\
&\leq \Big|\big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)\widehat{E}_1, \widehat{E}_2\big) \\
&\quad - \big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)\widehat{E}_1, \widehat{E}_2\big)^h\Big| \\
&\quad + \big|\big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)\widehat{E}_1, \widehat{E}_2\big)\big| \\
&\quad + \big|\big((u_1^h + 1)^2 \widehat{E}_2^-, \widehat{E}_2\big) - \big((u_1^h + 1)^2 \widehat{E}_2^-, \widehat{E}_2\big)^h\big| + \big|\big((u_1^h + 1)^2 \widehat{E}_2^-, \widehat{E}_2\big)\big| \\
&\leq \left| \int_\Omega (I - \pi^h)\big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)\widehat{E}_1 \widehat{E}_2\big)\, dx \right| \\
&\quad + \big|\big((U_2^n + U_2^{n-1} + 2)(U_1^n + u_1^h + 2)\widehat{E}_1, \widehat{E}_2\big)\big| \\
&\quad + \left| \int_\Omega (I - \pi^h)\big((u_1^h + 1)^2 \widehat{E}_2^- \widehat{E}_2\big)\, dx \right| + \big|\big((u_1^h + 1)^2 \widehat{E}_2^-, \widehat{E}_2\big)\big| \\
&= I_{6,2,1} + I_{6,2,2} + I_{6,2,3} + I_{6,2,3}.
\end{aligned} \tag{4.4.18}
$$

Now we bound each of the terms on the right hand side of (4.4.17) in turn. Noting (3.1.43), (4.2.42), (3.2.8a), the Poincaré inequality (2.1.2), and the Young inequality (2.1.4) we have

$$
\begin{aligned}
I_{5,2,1} &= \left| \int_\Omega (I - \pi^h)\big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^- \widehat{E}_1\big)\, dx \right| \\
&= \Big| \int_\Omega (I - \pi^h)\big[\big(U_1^n U_2^{n-1} + U_1^n(u_2^h + 2) + (U_1^{n-1} + 2)U_2^{n-1} \\
&\qquad + (U_1^{n-1} + 2)(u_2^h + 2)\big)\widehat{E}_2^- \widehat{E}_1\big]\, dx \Big| \\
&\leq C h^{2-d/3}\big(\|U_1^n\|_1\, \|U_2^{n-1}\|_1 + \|U_1^n\|_1\, \|u_2^h + 2\|_1 + \|U_1^{n-1} + 2\|_1\, \|U_2^{n-1}\|_1 \\
&\qquad + \|U_1^{n-1} + 2\|_1\, \|u_2^h + 2\|_1\big)\|\widehat{E}_2^-\|_1\, \|\widehat{E}_1\|_1 \\
&\leq C h^{2-d/3}\|\widehat{E}_2^-\|_1\, \|\widehat{E}_1\|_1, \\
&\leq C h^{4-2d/3}|\widehat{E}_1|_1, \\
&\leq C h^{4-2d/3} + \frac{1}{\epsilon}|\widehat{E}_1|_1^2,
\end{aligned} \tag{4.4.19}
$$

and

$$I_{5,2,3} = \left| \int_\Omega (I - \pi^h)\big((u_2^h + 1)^2 \widehat{E}_1^- \widehat{E}_1\big) dx \right| \le Ch^{4-2d/3} + \frac{1}{\epsilon}|\widehat{E}_1|_1^2. \tag{4.4.20}$$

Note that using the generalised Hölder inequality and (2.1.8) we have

$$(\chi_1\, \chi_2\, \chi_3\, \chi_4, 1) \le |\chi_1|_{0,6}\, |\chi_2|_{0,6}\, |\chi_3|_0\, |\chi_4|_{0,6} \le C\|\chi_1\|_1\, \|\chi_2\|_1\, |\chi_3|_0\, \|\chi_4\|_1.$$

Hence noting (4.2.42), (3.2.8a), the Poincaré inequality (2.1.2) and (4.4.13) we obtain

$$\begin{aligned}
I_{5,2,2} &\le \big|\big((U_1^n + U_1^{n-1} + 2)(U_2^{n-1} + u_2^h + 2)\widehat{E}_2^-, \widehat{E}_1\big)\big| \\
&\le C\big|\big(\big((U_1^n)^2 + (U_1^{n-1})^2 + (U_2^{n-1})^2 + (u_2^h)^2 + 4\big)|\widehat{E}_2^-|, |\widehat{E}_1|\big)\big| \\
&\le C\big(\|U_1^n\|_1^2 + \|U_1^{n-1}\|_1^2 + \|U_2^{n-1}\|_1^2 + \|u_2^h\|_1^2\big)|\widehat{E}_2^-|_0\, \|\widehat{E}_1\|_1 + C|\widehat{E}_2^-|_0\, |\widehat{E}_1|_0 \\
&\le C|\widehat{E}_2^-|_0\, |\widehat{E}_1|_1 \\
&\le C\big(|\widehat{E}_2^- - \widehat{E}_2|_0 + |\widehat{E}_2|_0\big)\, |\widehat{E}_1|_1 \\
&\le C\big(C(\Delta t)^2\Big|\frac{\partial U_2^n}{\partial t}\Big|_1 + |\widehat{E}_2|_0\big)\, |\widehat{E}_1|_1 \\
&\le C(\Delta t)^2\Big|\frac{\partial U_2^n}{\partial t}\Big|_1^2 + C\|E_2\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_1|_1^2,
\end{aligned} \tag{4.4.21}$$

and

$$I_{5,2,4} \le C(\Delta t)^2\Big|\frac{\partial U_1^n}{\partial t}\Big|_1^2 + C\|E_1\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_1|_1^2. \tag{4.4.22}$$

Using the same technique to bound $I_{5,2,k}$, $k = 1, \dots, 4$, we obtain

$$I_{6,2,1} \le Ch^{4-2d/3} + \frac{1}{\epsilon}\|\widehat{E}_2\|_1^2, \tag{4.4.23}$$

$$I_{6,2,3} \le Ch^{4-2d/3} + \frac{1}{\epsilon}\|\widehat{E}_2\|_1^2, \tag{4.4.24}$$

$$I_{6,2,2} \le C(\Delta t)^2\Big|\frac{\partial U_1^n}{\partial t}\Big|_1^2 + C\|E_1\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_2|_1, \tag{4.4.25}$$

$$I_{6,2,4} \le C(\Delta t)^2\Big|\frac{\partial U_2^n}{\partial t}\Big|_1^2 + C\|E_2\|_{-h}^2 + \frac{2}{\epsilon}|\widehat{E}_2|_1. \tag{4.4.26}$$

Substituting (4.4.7–4.4.10), (4.4.15–4.4.16), (4.4.19–4.4.26) into (4.4.6) we rewrite (4.4.6) as

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\big)+\gamma\big(|\widehat{E}_1|_1^2+|\widehat{E}_2|_1^2\big)\\
&\quad\le Ch^4\Big(\Big\|\widehat{\mathcal{G}}^h\frac{\partial U_1}{\partial t}\Big\|_1^2+\Big\|\widehat{\mathcal{G}}^h\frac{\partial U_2}{\partial t}\Big\|_1^2\Big)\\
&\qquad+C(h^4+(\Delta t)^2)\Big(\Big|\frac{\partial U_1}{\partial t}\Big|_1^2+\Big|\frac{\partial U_2}{\partial t}\Big|_1^2\Big)+\frac{(4+6D)}{\epsilon}\big(|\widehat{E}_1|_1^2+|\widehat{E}_2|_1^2\big)\\
&\qquad+C\Big(\Big\|\frac{\partial E_1}{\partial t}\Big\|_{-h}\|U_1-\widehat{U}_1\|_{-h}+\Big\|\frac{\partial E_2}{\partial t}\Big\|_{-h}\|U_2-\widehat{U}_2\|_{-h}\Big)\\
&\qquad+C(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2)+Ch^{4-2d/3}.
\end{aligned}
$$

Taking $\epsilon=(16+24D)/(3\gamma)$, simplifying and integrating over $t\in(0,T)$ we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\big(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2\big)ds+\frac{\gamma}{4}\int_0^T\big(|\widehat{E}_1|_1^2+|\widehat{E}_2|_1^2\big)ds\\
&\quad\le Ch^4\int_0^T\Big(\Big\|\widehat{\mathcal{G}}^h\frac{\partial U_1}{\partial t}\Big\|_1^2+\Big\|\widehat{\mathcal{G}}^h\frac{\partial U_2}{\partial t}\Big\|_1^2\Big)ds\\
&\qquad+C(h^4+(\Delta t)^2)\int_0^T\Big(\Big|\frac{\partial U_1}{\partial t}\Big|_1^2+\Big|\frac{\partial U_2}{\partial t}\Big|_1^2\Big)ds\\
&\qquad+C\int_0^T\Big(\Big\|\frac{\partial E_1}{\partial t}\Big\|_{-h}\|U_1-\widehat{U}_1\|_{-h}+\Big\|\frac{\partial E_2}{\partial t}\Big\|_{-h}\|U_2-\widehat{U}_2\|_{-h}\Big)ds\\
&\qquad+C\int_0^T(\|E_1\|_{-h}^2+\|E_2\|_{-h}^2)ds+C(T)h^{4-2d/3}.
\end{aligned}
$$

Using a Grönwall inequality, the Hölder inequality (2.1.7), the Poincaré inequality (3.1.3) and rearranging the terms we have

$$
\begin{aligned}
&\big(\|E_1(T)\|_{-h}^2+\|E_2(T)\|_{-h}^2\big)+\int_0^T\big(|\widehat{E}_1|_1^2+|\widehat{E}_2|_1^2\big)ds\\
&\le C\big(\|E_1(0)\|_{-h}^2+\|E_2(0)\|_{-h}^2\big)+Ch^4\int_0^T\Big(\Big|\widehat{\mathcal{G}}^h\frac{\partial U_1}{\partial t}\Big|_1^2+\Big|\widehat{\mathcal{G}}^h\frac{\partial U_2}{\partial t}\Big|_1^2\Big)ds\\
&\quad+C(h^4+(\Delta t)^2)\int_0^T\Big(\Big|\frac{\partial U_1}{\partial t}\Big|_1^2+\Big|\frac{\partial U_2}{\partial t}\Big|_1^2\Big)ds\\
&\quad+C\Big(\int_0^T\Big\|\frac{\partial E_1}{\partial t}\Big\|_{-h}^2ds\Big)^{1/2}\Big(\int_0^T\|U_1-\widehat{U}_1\|_{-h}^2ds\Big)^{1/2}\\
&\quad+C\Big(\int_0^T\Big\|\frac{\partial E_2}{\partial t}\Big\|_{-h}^2ds\Big)^{1/2}\Big(\int_0^T\|U_1-\widehat{U}_1\|_{-h}^2ds\Big)^{1/2}+C(T)h^{4-2d/3}. \qquad (4.4.27)
\end{aligned}
$$

On noting (4.4.1) and Theorem 4.2.1 we obtain

$$\int_0^T \left|\frac{\partial U_i}{\partial t}\right|_1^2 ds = \int_0^T \left|\frac{U_i^n - U_i^{n-1}}{\Delta t}\right|_1^2 ds = \sum_{n=1}^N \int_{t^{n-1}}^{t^n} \left|\frac{U_i^n - U_i^{n-1}}{\Delta t}\right|_1^2 ds$$
$$= \frac{1}{(\Delta t)^2} \sum_{n=1}^N \int_{t^{n-1}}^{t^n} |U_i^n - U_i^{n-1}|_1^2 ds \leq \frac{C}{\Delta t}. \tag{4.4.28}$$

The equations (3.2.1c), (3.2.1f), (4.2.1c), (4.2.1f) and (4.4.1) imply that

$$\|E_i(0)\|_{-h}^2 = \|u_i^h(0) - U_i^0\|_{-h}^2 = 0. \tag{4.4.29}$$

Noting (3.1.5), (4.1.6) and (4.2.7) we obtain

$$\int_0^T \|U_i - \widehat{U}_i\|_{-h}^2 ds \equiv \sum_{n=1}^N \int_{t^{n-1}}^{t^n} (t^n - t)\left\|\frac{\partial U_i}{\partial t}\right\|_{-h}^2 ds \leq (\Delta t)^2 \int_0^T \left|\mathcal{G}^h \frac{\partial U_i}{\partial t}\right|_1^2 ds$$
$$\leq C(\Delta t)^2 \sum_{n=1}^N \int_{t^{n-1}}^{t^n} (t^n - t)\left|\widehat{\mathcal{G}}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}]\right|_1^2 ds$$
$$\equiv C\Delta t \sum_{n=1}^N \left|\widehat{\mathcal{G}}^h[U_i^n - U_i^{n-1}]\right|_1^2 \leq C(\Delta t)^2. \tag{4.4.30}$$

Using (4.4.1), (3.1.11), (3.1.5), (3.2.8b) and (4.4.30) we obtain

$$\int_0^T \left\|\frac{\partial E_i}{\partial t}\right\|_{-h}^2 ds \leq 2\int_0^T \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-h}^2 ds + 2\int_0^T \left\|\frac{U_i^n - U_i^{n-1}}{\Delta t}\right\|_{-h}^2$$
$$\leq C\int_0^T \left\|\frac{\partial u_i^h}{\partial t}\right\|_{-1}^2 ds + C\int_0^T \left\|\mathcal{G}^h[\frac{U_i^n - U_i^{n-1}}{\Delta t}]\right\|_1^2 \leq C. \tag{4.4.31}$$

Hence noting (3.1.11), and (4.4.28–4.4.31) yields the desired result (4.4.5). □

Now we state a theorem to estimate the difference between the solutions u_i of the coupled pair of Cahn-Hilliard equations (2.2.1a–f) and their fully discrete approximation U_i^n defined by (4.2.1a–f):

Theorem 4.4.2 Let the assumptions of Theorem (4.2.1) hold. Then we have that for $i = 1, 2$,

$$\|u_i - U_i\|_{L^\infty(0,T;H^1(\Omega)')} + \|u_i - \widehat{U}_i\|_{L^2(0,T;H^1(\Omega))} \leq C\Big[\Delta t + h^2 + \frac{h^4}{\Delta t}\Big]. \tag{4.4.32}$$

Proof. The result follows from combining (3.3.1) and (4.4.5). □

Corollary 4.4.3 Let the assumptions of Theorem 4.4.2 hold and $\Delta t = \mathcal{O}(h^2)$. Then this gives

$$\|u_i - U_i\|_{L^\infty(0,T;H^1(\Omega)')} + \|u_i - \widehat{U}_i\|_{L^2(0,T;H^1(\Omega))} \leq Ch^2,$$

which is optimal in $H^1(\Omega)$.

Proof. It follows from Theorem 4.4.2. □

Chapter 5

Numerical Experiments

In this chapter we discuss two practical algorithms (implicit and explicit method) that are used to solve an algebraic system arising from the problem discussed in this thesis. We discuss the convergence theory for the implicit scheme used to solve the system arising from Scheme 1. We also discuss some computational results for one and two dimensions. We used the implicit scheme for all simulations in this chapter. We have made a comparison with Scheme 2 and the results are similar. Before showing some computational results, we discuss the linear stability solution for the problem.

5.1 Practical Algorithms

5.1.1 Iterative Method for Scheme 1

Let us expand U_i and W_i, $i = 1, 2$, in terms of the standard nodal basis functions of the finite element space S^h, that is,

$$U_1^n = \sum_{i=1}^{J} U_{1,i}^n \eta_i, \quad W_1^n = \sum_{i=1}^{J} W_{1,i}^n \eta_i, \tag{5.1.1a}$$

$$U_2^n = \sum_{i=1}^{J} U_{2,i}^n \eta_i, \quad W_2^n = \sum_{i=1}^{J} W_{2,i}^n \eta_i, \tag{5.1.1b}$$

where J be the number of node points.

Given $\boldsymbol{y} \in S := \{\boldsymbol{y} \in \mathbb{R}^J : \mathbf{1}^t M \boldsymbol{y} = 0\}$. The existence of $\widehat{\mathcal{G}}^h$ defines, implicitly, the invertible linear transformation $T : S \mapsto S$ by

$$T(\boldsymbol{y}) = \widetilde{\boldsymbol{y}},$$

where $\widetilde{\boldsymbol{y}}$ is the solution of

$$K\widetilde{\boldsymbol{y}} = M\boldsymbol{y},$$
$$\mathbf{1}^t M\widetilde{\boldsymbol{y}} = 0,$$

where $\mathbf{1}$ is a vector with components 1, M is a mass matrix and K is a stiffness matrix. That is $M^{-1}KT(\boldsymbol{y}) = \boldsymbol{y}$ and $T(M^{-1}K\widetilde{\boldsymbol{y}}) = \widetilde{\boldsymbol{y}}$. Also note that $\mathbf{1}^t MT(\boldsymbol{y}) = \mathbf{1}^t M\widetilde{\boldsymbol{y}} = 0$.

Now substituting (5.1.1a–b) into (4.2.6a) and noting (4.2.5) we have for a jth element of the basis function $\{\eta_j\}$ the following:

$$\Big(\widehat{\mathcal{G}}^h \sum_{i=1}^{J}\Big[\frac{U_{1,i}^n - U_{1,i}^{n-1}}{\Delta t}\Big]\eta_i, \eta_j\Big)^h + \gamma\Big(\sum_{i=1}^{J} U_{1,i}^n \nabla\eta_i, \nabla\eta_j\Big)$$
$$+\sum_{i=1}^{J}\Big((\eta_i,\eta_j)^h F_1(U_{1,i},U_{2,i}) - \frac{1}{|\Omega|}(\eta_i,1)^h F_1(U_{1,i},U_{2,i})(1,\eta_j)^h\Big) = 0,$$

which is equivalent to

$$MT\Big(\frac{\boldsymbol{U}_1^n - \boldsymbol{U}_1^{n-1}}{\Delta t}\Big) + \gamma K\boldsymbol{U}_1^n + MF_1(\boldsymbol{U}_1^n, \boldsymbol{U}_2^n) - \lambda_1^n M\mathbf{1} = 0,$$
$$\iff T\Big(\frac{\boldsymbol{U}_1^n - \boldsymbol{U}_1^{n-1}}{\Delta t}\Big) + \gamma M^{-1}K\boldsymbol{U}_1^n + F_1(\boldsymbol{U}_1^n, \boldsymbol{U}_2^n) - \lambda_1^n\mathbf{1} = 0,$$
$$\iff T\Big(\frac{\boldsymbol{U}_1^n - \boldsymbol{U}_1^{n-1}}{\Delta t}\Big) + \gamma T(M^{-1}KM^{-1}K\boldsymbol{U}_1^n) + F_1(\boldsymbol{U}_1^n, \boldsymbol{U}_2^n) - \lambda_1^n\mathbf{1} = 0, \quad (5.1.2a)$$

and similarly

$$T\Big(\frac{\boldsymbol{U}_2^n - \boldsymbol{U}_2^{n-1}}{\Delta t}\Big) + \gamma T(M^{-1}KM^{-1}K\boldsymbol{U}_2^n) + F_2(\boldsymbol{U}_1^n, \boldsymbol{U}_2^n) - \lambda_2^n\mathbf{1} = 0, \qquad (5.1.2b)$$

where for $i = 1, 2$,

$$\lambda_i^n = \frac{\mathbf{1}^t M F_i(\boldsymbol{U}_1^n, \boldsymbol{U}_2^n)}{|\Omega|}. \tag{5.1.2c}$$

Our aim is to solve the algebraic nonlinear systems (5.1.2a–b). To accomplish this let us define the operators $\mathcal{A}_1$,$\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ such that

$$\begin{aligned}
\mathcal{A}_1 &: (a,b)^J \mapsto \mathbb{R}^J & \mathcal{A}_1(\boldsymbol{y}) &= F_1(\boldsymbol{y}, \boldsymbol{y}_2),\\
\mathcal{A}_2 &: (a,b)^J \mapsto \mathbb{R}^J & \mathcal{A}_2(\boldsymbol{y}) &= F_2(\boldsymbol{y}_1, \boldsymbol{y}),\\
\mathcal{B}_1 &: S_1^h \mapsto S & \mathcal{B}_1(\boldsymbol{y}_1) &= T\Big(\tfrac{\boldsymbol{y}_1 - \boldsymbol{U}_1^{n-1}}{\Delta t}\Big) + \gamma T(M^{-1}KM^{-1}K\boldsymbol{y}_1),\\
\mathcal{B}_2 &: S_2^h \mapsto S & \mathcal{B}_2(\boldsymbol{y}_2) &= T\Big(\tfrac{\boldsymbol{y}_2 - \boldsymbol{U}_2^{n-1}}{\Delta t}\Big) + \gamma T(M^{-1}KM^{-1}K\boldsymbol{y}_2),
\end{aligned}$$

where

$$\begin{aligned}
S_1^h &:= \{\boldsymbol{y}_1 \in \mathbb{R}^J : \mathbf{1}^t M \boldsymbol{y}_1 = (u_{1,0}^h, 1)^h\},\\
S_2^h &:= \{\boldsymbol{y}_2 \in \mathbb{R}^J : \mathbf{1}^t M \boldsymbol{y}_2 = (u_{2,0}^h, 1)^h\},
\end{aligned}$$

so that (5.1.2a–b) can be written as:

$$\mathcal{B}_1(\boldsymbol{U}_1^n) + \mathcal{A}_1(\boldsymbol{U}_1^n) - \lambda_1^n \mathbf{1} = 0, \tag{5.1.3a}$$

$$\mathcal{B}_2(\boldsymbol{U}_2^n) + \mathcal{A}_2(\boldsymbol{U}_2^n) - \lambda_2^n \mathbf{1} = 0. \tag{5.1.3b}$$

To solve the systems (5.1.3a–b) we adapt the algorithm of Lions and Meisier [28], who consider the case where $\mathcal{A}_i$ and $\mathcal{B}_i$ are two general maximal monotone operators in the absence of Lagrange multipliers. Copetti and Elliott [16] have adapted this algorithm where there is single Lagrange multiplier present. Barrett and Blowey [3] have also adapted this algorithm when considering the finite element approximation of a model for the phase separation of a multi-component alloy with non-smooth free energy, where there are two Lagrange multipliers present.

Multiplying both sides of (5.1.3a) with $\mu > 0$, adding $\boldsymbol{U}_1^n$ to both sides, and rearranging the terms we have

$$U_1^n + \mu\mathcal{A}_1(U_1^n) = U_1^n - \mu\mathcal{B}_1(U_1^n) + \mu\lambda_1^n\mathbf{1}. \tag{5.1.4a}$$

In the same way from (5.1.3b) we obtain

$$U_2^n + \mu\mathcal{A}_2(U_2^n) = U_2^n - \mu\mathcal{B}_2(U_2^n) + \mu\lambda_2^n\mathbf{1}. \tag{5.1.4b}$$

Now define

$$Z_1^n = U_1^n - \mu\mathcal{B}_1(U_1^n) + \mu\lambda_1^n\mathbf{1}, \tag{5.1.5a}$$

$$Z_2^n = U_2^n - \mu\mathcal{B}_2(U_2^n) + \mu\lambda_2^n\mathbf{1}, \tag{5.1.5b}$$

$$X_1^n = 2U_1^n - Z_1^n = U_1^n + \mu\mathcal{B}_1(U_1^n) - \mu\lambda_1^n\mathbf{1}, \tag{5.1.5c}$$

$$X_2^n = 2U_2^n - Z_2^n = U_2^n + \mu\mathcal{B}_2(U_2^n) - \mu\lambda_2^n\mathbf{1}. \tag{5.1.5d}$$

We perform the iteration as follows:
Find $U_1^{n,j+\frac{1}{2}}$ such that

$$U_1^{n,j+\frac{1}{2}}(x_i) + \mu\mathcal{A}_1(U_1^{n,j+\frac{1}{2}}(x_i)) = Z_{1,i}^{n,j} \quad \text{for} \quad i = 1, \dots, J, \tag{5.1.6a}$$

and set

$$X_1^{n,j+1} = 2U_1^{n,j+\frac{1}{2}} - Z_1^{n,j}. \tag{5.1.6b}$$

Then find $U_1^{n,j+1}$ and $U_2^{n,j+1}$ such that

$$U_1^{n,j+1} + \mu\mathcal{B}_1(U_1^{n,j+1}) - \mu\lambda_1^{n,j+1}\mathbf{1} = X_1^{n,j+1}, \tag{5.1.6c}$$

where

$$\lambda_1^{n,j+1} = \frac{1}{\mu\mathbf{1}^t M\mathbf{1}}\left((u_1^{0,h}, 1)^h - \mathbf{1}^t M X_1^{n,j+1}\right), \tag{5.1.6d}$$

and set

$$Z_1^{n,j+1} = 2U_1^{n,j+1} - X_1^{n,j+1}. \tag{5.1.6e}$$

Next find $\boldsymbol{U}_2^{n,j+\frac{1}{2}}$ such that

$$\boldsymbol{U}_2^{n,j+\frac{1}{2}}(x_i) + \mu\mathcal{A}_2(\boldsymbol{U}_2^{n,j+\frac{1}{2}}(x_i)) = \boldsymbol{Z}_{2,i}^{n,j} \quad \text{for} \quad i = 1, \dots, J, \tag{5.1.7a}$$

and set

$$\boldsymbol{X}_2^{n,j+1} = 2\boldsymbol{U}_2^{n,j+\frac{1}{2}} - \boldsymbol{Z}_2^{n,j}. \tag{5.1.7b}$$

Finally, find $\boldsymbol{U}_2^{n,j+1}$ such that

$$\boldsymbol{U}_2^{n,j+1} + \mu\mathcal{B}_2(\boldsymbol{U}_2^{n,j+1}) - \mu\lambda_2^{n,j+1}\mathbf{1} = \boldsymbol{X}_2^{n,j+1}, \tag{5.1.7c}$$

where

$$\lambda_2^{n,j+1} = \frac{1}{\mu\mathbf{1}^t M\mathbf{1}}\left((u_2^{0,h}, 1)^h - \mathbf{1}^t M\boldsymbol{X}_2^{n,j+1}\right), \tag{5.1.7d}$$

and set

$$\boldsymbol{Z}_2^{n,j+1} = 2\boldsymbol{U}_2^{n,j+1} - \boldsymbol{X}_2^{n,j+1}. \tag{5.1.7e}$$

Notice that $\{\boldsymbol{U}_1^{n,j}\}$ and $\{\boldsymbol{U}_2^{n,j}\}$ are independent sequences.

Lemma 5.1.1 The operators $\mathcal{A}_i$ are maximal monotone and $\mathcal{B}_i$ are coercive.

Proof. We show monotonicity of $\mathcal{A}_i$ by showing the monotonicity of F_i with y_j fixed, $i \neq j$. Consider F_1, i.e.

$$\begin{aligned}
F_1(y, y_2) &= y^3 - y_1^{(n-1)} + D(y + y_1^{(n-1)} + 2)(y_2 + 1)^2 \\
&= y^3 + Dy(y_2 + 1)^2 + 2D(y_2 + 1)^2 + (D(y_2 + 1)^2 - 1)y_1^{(n-1)} \\
&= y^3 + c_1 y + c_2,
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= D(y_2 + 1)^2 \geq 0, \\
c_2 &= 2D(y_2 + 1)^2 + (D(y_2 + 1)^2 - 1)y_1^{(n-1)}.
\end{aligned}$$

Now without loss of generality let $0 < a < b$, then

$$F_1(a, y_2) = a^3 + c_1 a + c_2, \tag{5.1.8}$$

$$F_1(b, y_2) = b^3 + c_1 b + c_2. \tag{5.1.9}$$

Subtracting (5.1.8) from (5.1.9) we have $F_1(b, y_2) \geq F_1(a, y_2)$. Similarly we can show that $F_2(y_1, b) \geq F_2(y_1, a)$. Since the range of $I + \mu \mathcal{A}_i \in \mathbb{R}^J$, then $\mathcal{A}_i$ is maximal (see Zeidler [36] page 843).

To show $\mathcal{B}_i$ are coercive, given $\boldsymbol{v}_i, \boldsymbol{w}_i \in S_i^h$, define $(\boldsymbol{v}_i, \boldsymbol{w}_i) = \boldsymbol{w}_i^t M \boldsymbol{v}_i$, where $(\cdot, \cdot)$ is an inner product on $\mathbb{R}^J$. Then, for i fixed,

$$\begin{aligned}
&\big(\mathcal{B}_i(\boldsymbol{w}_i) - \mathcal{B}_i(\boldsymbol{v}_i), \boldsymbol{w}_i - \boldsymbol{v}_i\big) \\
&\quad = \left(T\Big(\frac{\boldsymbol{w}_i - \boldsymbol{U}_i^{n-1}}{\Delta t}\Big) + \gamma M^{-1} K \boldsymbol{w}_i - T\Big(\frac{\boldsymbol{v}_i - \boldsymbol{U}_i^{n-1}}{\Delta t}\Big) - \gamma M^{-1} K \boldsymbol{v}_i, \boldsymbol{w}_i - \boldsymbol{v}_i\right) \\
&\quad = \left(T\Big(\frac{\boldsymbol{w}_i - \boldsymbol{v}_i}{\Delta t}\Big) + \gamma M^{-1} K (\boldsymbol{w}_i - \boldsymbol{v}_i), \boldsymbol{w}_i - \boldsymbol{v}_i\right) \\
&\quad = \left(T\Big(\frac{\boldsymbol{w}_i - \boldsymbol{v}_i}{\Delta t}\Big), \boldsymbol{w}_i - \boldsymbol{v}_i\right) + \big(\gamma M^{-1} K (\boldsymbol{w}_i - \boldsymbol{v}_i), \boldsymbol{w}_i - \boldsymbol{v}_i\big) \\
&\quad = \frac{1}{\Delta t} (\boldsymbol{w}_i - \boldsymbol{v}_i)^t M T (\boldsymbol{w}_i - \boldsymbol{v}_i) + \gamma (\boldsymbol{w}_i - \boldsymbol{v}_i)^t K (\boldsymbol{w}_i - \boldsymbol{v}_i).
\end{aligned} \tag{5.1.10}$$

Define $\boldsymbol{\chi}_i = \sum_{j=1}^J w_{i,j} \eta_j$ and $\boldsymbol{\nu}_i = \sum_{k=1}^J v_{i,k} \eta_k$ and dropping the index i, we have

$$\gamma |\boldsymbol{\chi} - \boldsymbol{\nu}|_1^2 = \gamma (\boldsymbol{w} - \boldsymbol{v})^t K (\boldsymbol{w} - \boldsymbol{v}),$$

and on noting $\boldsymbol{\chi} - \boldsymbol{\nu} \in S$

$$\begin{aligned}
\frac{1}{\Delta t} |\widehat{\mathcal{G}}^h (\boldsymbol{\chi} - \boldsymbol{\nu})|_1^2 &= (\nabla \widehat{\mathcal{G}}^h (\boldsymbol{\chi} - \boldsymbol{\nu}), \nabla \widehat{\mathcal{G}}^h (\boldsymbol{\chi} - \boldsymbol{\nu})) \\
&= \frac{1}{\Delta t} (\widehat{\mathcal{G}}^h (\boldsymbol{\chi} - \boldsymbol{\nu}), (\boldsymbol{\chi} - \boldsymbol{\nu}))^h \\
&= \frac{1}{\Delta t} \left(\widehat{\mathcal{G}}^h \sum_{j=1}^J (w_j - v_j) \eta_j, \sum_{k=1}^J (w_k - v_k) \eta_k \right)^h \\
&= \frac{1}{\Delta t} (\boldsymbol{w} - \boldsymbol{v})^t M T (\boldsymbol{w} - \boldsymbol{v}).
\end{aligned}$$

Hence we can rewrite (5.1.10) as

$$\big(\mathcal{B}(\boldsymbol{w}) - \mathcal{B}(\boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v}\big) = \frac{1}{\Delta t}|\widehat{\mathcal{G}}^h(\boldsymbol{\chi} - \boldsymbol{\nu})|_1^2 + \gamma|\boldsymbol{\chi} - \boldsymbol{\nu}|_1^2 \geq \gamma|\boldsymbol{\chi} - \boldsymbol{\nu}|_1^2.$$

Noting the Poincaré inequality (3.1.3), we have

$$\big(\mathcal{B}(\boldsymbol{w}) - \mathcal{B}(\boldsymbol{v}), \boldsymbol{w} - \boldsymbol{v}\big) \geq C|\boldsymbol{\chi} - \boldsymbol{\nu}|_h^2 \geq C(\boldsymbol{w} - \boldsymbol{v})^t M(\boldsymbol{w} - \boldsymbol{v}),$$

and therefore $\mathcal{B}_i$ are coercive. □

To see how we can compute $\boldsymbol{U}_1^{n,j+\frac{1}{2}}$ from (5.1.6a), expand $\mathcal{A}_1\boldsymbol{U}_1^{n,j+\frac{1}{2}}$ about the ith component ($i = 1, \ldots, J$) as follows:

$$\begin{aligned}
\mathcal{A}_1(\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i)) =& F_1(\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i), \boldsymbol{U}_2^{n-1}(x_i)) \\
=& (\boldsymbol{U}_1^{n,j+\frac{1}{2}})^3(x_i) - \boldsymbol{U}_1^{n-1}(x_i) \\
&+ D(\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i) + \boldsymbol{U}_1^{n-1}(x_i) + 2)(\boldsymbol{U}_2^{n-1}(x_i) + 1)^2.
\end{aligned}$$

So the equation (5.1.6a) can be written as

$$\begin{aligned}
\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i) + \mu\Big(&(\boldsymbol{U}_1^{n,j+\frac{1}{2}})^3(x_i) - \boldsymbol{U}_1^{n-1}(x_i) \\
&+ D\big(\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i) + \boldsymbol{U}_1^{n-1}(x_i) + 2\big)(\boldsymbol{U}_2^{n-1}(x_i) + 1)^2\Big) = \boldsymbol{Z}_{1,i}^{n,j}.
\end{aligned}$$

Thus to find $\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i)$, ($i = 1, \ldots, J$,) we solve the following equation

$$\begin{aligned}
\big(1+&\mu D(\boldsymbol{U}_2^{n-1}(x_i) + 1)^2\big)\boldsymbol{U}_1^{n,j+\frac{1}{2}}(x_i) + \mu(\boldsymbol{U}_1^{n,j+\frac{1}{2}})^3(x_i) \\
&= \boldsymbol{Z}_{1,i}^{n,j} + \mu\boldsymbol{U}_1^{n-1}(x_i) - \mu D\big(\boldsymbol{U}_1^{n-1}(x_i) + 2\big)(\boldsymbol{U}_2^{n-1}(x_i) + 1)^2.
\end{aligned} \tag{5.1.11a}$$

Similarly, we find $\boldsymbol{U}_2^{n,j+\frac{1}{2}}(x_i)$, ($i = 1, \ldots, J$,) from

$$\begin{aligned}
\big(1+&\mu D(\boldsymbol{U}_1^{n}(x_i) + 1)^2\big)\boldsymbol{U}_2^{n,j+\frac{1}{2}}(x_i) + \mu(\boldsymbol{U}_2^{n,j+\frac{1}{2}})^3(x_i) \\
&= \boldsymbol{Z}_{2,i}^{n,j} + \mu\boldsymbol{U}_2^{n-1}(x_i) - \mu D\big(\boldsymbol{U}_2^{n-1}(x_i) + 2\big)(\boldsymbol{U}_1^{n}(x_i) + 1)^2.
\end{aligned} \tag{5.1.11b}$$

Now let us discuss how to obtain $\boldsymbol{U}_1^{n,j+1}$ from (5.1.6c). Since 0 is a simple eigenvalue of K with eigenvector $\mathbf{1}$, solving (5.1.6c) is equivalent to:

$$\begin{aligned}
K\boldsymbol{X}_1^{nj+1} &= K\boldsymbol{U}_1^{n,j+1} + \mu K\mathcal{B}_1(\boldsymbol{U}_1^{n,j+1}) \\
&= K\boldsymbol{U}_1^{n,j+1} + \mu K\left(T\left(\frac{\boldsymbol{U}_1^{n,j+1} - \boldsymbol{U}_1^{n-1}}{\Delta t}\right) + \gamma M^{-1}K\boldsymbol{U}_1^{n,j+1}\right) \\
&= K\boldsymbol{U}_1^{n,j+1} + \mu M\left(\frac{\boldsymbol{U}_1^{n,j+1} - \boldsymbol{U}_1^{n-1}}{\Delta t}\right) + \mu\gamma K M^{-1}K\boldsymbol{U}_1^{n,j+1},
\end{aligned}$$

which after rearranging gives

$$(\Delta t K + \mu M + \Delta t\mu\gamma K M^{-1}K)\boldsymbol{U}_1^{n,j+1} = \mu M\boldsymbol{U}_1^{n-1} + \Delta t K\boldsymbol{X}_1^{n,j+1}. \tag{5.1.12a}$$

Similarly solving the equation (5.1.7c) is equivalent to

$$(\Delta t K + \mu M + \Delta t\mu\gamma K M^{-1}K)\boldsymbol{U}_2^{n,j+1} = \mu M\boldsymbol{U}_2^{n-1} + \Delta t K\boldsymbol{X}_2^{n,j+1}. \tag{5.1.12b}$$

The matrix $\Delta t K + \mu M + \Delta t\mu\gamma K M^{-1}K$ is symmetric positive definite, therefore the system has a unique solution.

Theorem 5.1.2 For all $\mu \in \mathbb{R}^+$ and $\{\boldsymbol{U}_1^{n,0}, \lambda_1^{n,0}, \boldsymbol{U}_2^{n,0}, \lambda_2^{n,0}\} \in S_1^h \times \mathbb{R} \times S_2^h \times \mathbb{R}$, the sequences $\{\boldsymbol{U}_1^{n,j}\}_{j\geq 0}$ and $\{\boldsymbol{U}_2^{n,j}\}_{j\geq 0}$ generated by algorithms (5.1.6a–d) and (5.1.7a–d) converge to the unique solution of (5.1.3a) and (5.1.3b) respectively.

Proof. The proof is the same as that of Copetti and Elliott [16] and for completeness we repeat their proof. Since the algorithms are independent and similar we prove both of them at the same time. For notational convenience, we drop the dependence on the time level index n and component i, throughout this proof. We define (compare with (5.1.5a–d))

$$\boldsymbol{X} = \boldsymbol{U} + \mu\mathcal{B}(\boldsymbol{U}) - \mu\lambda\mathbf{1}, \tag{5.1.13}$$

$$\boldsymbol{Z} = \boldsymbol{U} + \mu\mathcal{A}(\boldsymbol{U}), \tag{5.1.14}$$

$$\boldsymbol{a} = \mathcal{A}(\boldsymbol{U}), \tag{5.1.15}$$

$$\boldsymbol{X}^j = \boldsymbol{U}^j + \mu\mathcal{B}(\boldsymbol{U}^j) - \mu\lambda^j\mathbf{1}, \tag{5.1.16}$$

$$\boldsymbol{U}^j = \frac{\boldsymbol{Z}^j + \boldsymbol{X}^j}{2}, \tag{5.1.17}$$

$$\boldsymbol{a}^j = \frac{\boldsymbol{Z}^j - \boldsymbol{X}^{j+1}}{2\mu}. \tag{5.1.18}$$

Adding (5.1.13) and (5.1.14), noting (5.1.3a–b), and rearranging the terms we have

$$\boldsymbol{U} = \frac{\boldsymbol{X} + \boldsymbol{Z}}{2}. \tag{5.1.19}$$

Subtracting (5.1.13) from (5.1.14), noting

$$\mathcal{A}(\boldsymbol{U}) = \lambda\mathbf{1} - \mathcal{B}(\boldsymbol{U}),$$

and simplifying we obtain

$$\boldsymbol{a} = \frac{\boldsymbol{Z} - \boldsymbol{X}}{2\mu}.$$

On substituting (5.1.6b) into (5.1.6c) and (5.1.7b) into (5.1.7c), noting (5.1.14) and (5.1.16), we have

$$\boldsymbol{X}^{j+1} = (I - \mu\mathcal{A})\boldsymbol{U}^{j+\frac{1}{2}}.$$

Hence noting (5.1.6a) and (5.1.7a), we can rewrite the iteration (5.1.6a–d) and (5.1.7a–d) as

$$\begin{aligned}\boldsymbol{X}^{j+1} &= (I - \mu\mathcal{A})(I + \mu\mathcal{A})^{-1}\boldsymbol{Z}^j \\ &= (2I - (I + \mu\mathcal{A}))(I + \mu\mathcal{A})^{-1}\boldsymbol{Z}^j,\end{aligned}$$

which implies

$$\mathcal{J}_{\mathcal{A}}^{\mu}(\boldsymbol{Z}^j) = \frac{\boldsymbol{X}^{j+1} - \boldsymbol{Z}^j}{2},$$

where $\mathcal{J}_{\mathcal{A}}^{\mu} = (I + \mu\mathcal{A})^{-1}$. Therefore

$$\begin{aligned}
\boldsymbol{X}^{j+1} &= (I - \mu\mathcal{A})\mathcal{J}_{\mathcal{A}}^{\mu}(\boldsymbol{Z}^j) \\
&= (I - \mu\mathcal{A})\left(\frac{\boldsymbol{X}^{j+1} - \boldsymbol{Z}^j}{2}\right) \\
&= \left(\frac{\boldsymbol{X}^{j+1} - \boldsymbol{Z}^j}{2}\right) - \mu\mathcal{A}\left(\frac{\boldsymbol{X}^{j+1} - \boldsymbol{Z}^j}{2}\right),
\end{aligned}$$

which implies

$$\boldsymbol{a}^j = \frac{\boldsymbol{Z}^j - \boldsymbol{X}^{j+1}}{2\mu} = \mathcal{A}\left(\frac{\boldsymbol{X}^{j+1} - \boldsymbol{Z}^j}{2}\right).$$

The monotonicity of $\mathcal{B}$ and the fact that $(\mathbf{1}, \boldsymbol{U}^j - \boldsymbol{U}) = 0$ yields

$$\begin{aligned}
0 &\le (\mathcal{B}(\boldsymbol{U}^j) - \mathcal{B}(\boldsymbol{U}), \boldsymbol{U}^j - \boldsymbol{U}) \\
&= \left(\frac{\boldsymbol{X}^j - \boldsymbol{U}^j}{\mu} + \lambda^j\mathbf{1} - \frac{\boldsymbol{X} - \boldsymbol{U}}{\mu} - \lambda\mathbf{1}, \boldsymbol{U}^j - \boldsymbol{U}\right) \\
&= \frac{1}{\mu}((\boldsymbol{X}^j - \boldsymbol{X}) - (\boldsymbol{U}^j - \boldsymbol{U}), \boldsymbol{U}^j - \boldsymbol{U}) + (\lambda^j - \lambda)(\mathbf{1}, \boldsymbol{U}^j - \boldsymbol{U}) \\
&= \frac{1}{\mu}\left(\frac{(\boldsymbol{X}^j - \boldsymbol{X})}{2} - \frac{(\boldsymbol{Z}^j - \boldsymbol{Z})}{2}, \frac{(\boldsymbol{X}^j - \boldsymbol{X})}{2} + \frac{(\boldsymbol{Z}^j - \boldsymbol{X})}{2}\right) \\
&= \frac{1}{4\mu}(|\boldsymbol{X}^j - \boldsymbol{X}|^2 - |\boldsymbol{Z}^j - \boldsymbol{Z}|^2),
\end{aligned} \tag{5.1.20}$$

where we have noted (5.1.17) and (5.1.19).

From the monotonicity of $\mathcal{A}$, we obtain

$$\begin{aligned}
0 &\le \left(\boldsymbol{a}^j - \boldsymbol{a}, \frac{(\boldsymbol{X}^{j+1} + \boldsymbol{Z}^j)}{2} - \boldsymbol{U}\right) \\
&= \left(\frac{(\boldsymbol{Z}^j + \boldsymbol{X}^{j+1})}{2\mu} - \frac{(\boldsymbol{Z} - \boldsymbol{X})}{2\mu}, \frac{(\boldsymbol{X}^{j+1} + \boldsymbol{Z}^j)}{2} - \frac{(\boldsymbol{Z} + \boldsymbol{X})}{2}\right) \\
&= \frac{1}{4\mu}((\boldsymbol{Z}^j - \boldsymbol{Z}) - (\boldsymbol{X}^{j+1} - \boldsymbol{X}), (\boldsymbol{Z}^j - \boldsymbol{Z}) + (\boldsymbol{X}^{j+1} - \boldsymbol{X})) \\
&= \frac{1}{4\mu}(|\boldsymbol{Z}^j - \boldsymbol{Z}|^2 - |\boldsymbol{X}^{j+1} - \boldsymbol{X}|^2).
\end{aligned} \tag{5.1.21}$$

From (5.1.20) and (5.1.21) we conclude that

$$|\boldsymbol{X}^{j+1} - \boldsymbol{X}|^2 \leq |\boldsymbol{Z}^j - \boldsymbol{Z}|^2 \leq |\boldsymbol{X}^j - \boldsymbol{X}|^2,$$

i.e. $\{|\boldsymbol{X}^j - \boldsymbol{X}|^2\}$ is a decreasing sequence that is bounded below so $|\boldsymbol{X}^j - \boldsymbol{X}|^2$ converges. Now, adding (5.1.20) and (5.1.21), we have

$$\begin{aligned}
0 &\leq (\mathcal{B}(\boldsymbol{U}^j) - \mathcal{B}(\boldsymbol{U}), \boldsymbol{U}^j - \boldsymbol{U}) + \left(\boldsymbol{a}^j - \boldsymbol{a}, \frac{(\boldsymbol{X}^{j+1} + \boldsymbol{Z}^j)}{2} - \boldsymbol{U}\right) \\
&\leq \frac{1}{4\mu}\left(|\boldsymbol{X}^j - \boldsymbol{X}|^2 - |\boldsymbol{X}^{j+1} - \boldsymbol{X}|^2\right),
\end{aligned}$$

which tends to zero as $j \to \infty$. On noting (5.1.21) this shows that

$$(\mathcal{B}(\boldsymbol{U}^j) - \mathcal{B}(\boldsymbol{U}), \boldsymbol{U}^j - \boldsymbol{U}) \to 0 \quad \text{as} \quad j \to \infty.$$

Since $\mathcal{B}$ is coercive, we conclude that $\boldsymbol{U}^j \to \boldsymbol{U}$ as $j \to \infty$. □

5.1.2 Scheme 2

We now show that Scheme 2 is linear. On substituting (5.1.1a–b) into (4.3.1a–f) and taking $\eta = \eta_j$ for $j = 1, \dots, J$ we have

$$\sum_{i=1}^{J} (\eta_i, \eta_j)^h (U_{1,i}^{n+1} - U_{1,i}^{n-1}) = -2\Delta t \sum_{i=1}^{J} (\nabla\eta_i, \nabla\eta_j) W_{1,i}^n, \tag{5.1.22a}$$

$$\sum_{i=1}^{J} (\eta_i, \eta_j)^h W_{1,i}^n = \sum_{i=1}^{J} (\eta_i, \eta_j)^h F_1(U_{1,i}^{n+1}, U_{2,i}^{n+1}) + \frac{\gamma}{2} \sum_{i=1}^{J} (\nabla\eta_i, \nabla\eta_j)(U_{1,i}^{n+1} + U_{1,i}^{n-1}), \tag{5.1.22b}$$

and

$$\sum_{i=1}^{J} (\eta_i, \eta_j)^h (U_{2,i}^{n+1} - U_{2,i}^{n-1}) = -2\Delta t \sum_{i=1}^{J} (\nabla\eta_i, \nabla\eta_j) W_{2,i}^n, \tag{5.1.22c}$$

$$\sum_{i=1}^{J} (\eta_i, \eta_j)^h W_{2,i}^n = \sum_{i=1}^{J} (\eta_i, \eta_j)^h F_2(U_{1,i}^{n+1}, U_{2,i}^{n+1}) + \frac{\gamma}{2} \sum_{i=1}^{J} (\nabla\eta_i, \nabla\eta_j)(U_{2,i}^{n+1} + U_{2,i}^{n-1}), \tag{5.1.22d}$$

where

$$F_1(U_{1,i}^{n+1}, U_{2,i}^{n+1}) = (U_{1,i}^n)^2 (\frac{U_{1,i}^{n+1} + U_{1,i}^{n-1}}{2}) - U_{1,i}^n + D(U_{1,i}^{n+1} + U_{1,i}^{n-1} + 2)(U_{2,i}^n + 1)^2, \tag{5.1.22e}$$

$$F_2(U_{1,i}^{n+1}, U_{2,i}^{n+1}) = (U_{2,i}^n)^2 (\frac{U_{2,i}^{n+1} + U_{2,i}^{n-1}}{2}) - U_{2,i}^n + D(U_{2,i}^{n+1} + U_{2,i}^{n-1} + 2)(U_{1,i}^n + 1)^2. \tag{5.1.22f}$$

The equations (5.1.22a–f) lead to the following systems:

$$\begin{aligned} M(\boldsymbol{U}_1^{n+1} - \boldsymbol{U}_1^{n-1}) &= -2\Delta t K \boldsymbol{W}_1^n \\ M\boldsymbol{W}_1^n &= M\boldsymbol{F}_1(\boldsymbol{U}_1^{n+1}, \boldsymbol{U}_2^{n+1}) + \frac{\gamma}{2} K(\boldsymbol{U}_1^{n+1} + \boldsymbol{U}_1^{n-1}), \end{aligned}$$

and

$$\begin{aligned} M(\boldsymbol{U}_2^{n+1} - \boldsymbol{U}_2^{n-1}) &= -2\Delta t K \boldsymbol{W}_2^n \\ M\boldsymbol{W}_2^n &= M\boldsymbol{F}_2(\boldsymbol{U}_1^{n+1}, \boldsymbol{U}_2^{n+1}) + \frac{\gamma}{2} K(\boldsymbol{U}_2^{n+1} + \boldsymbol{U}_2^{n-1}), \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{U}_1^n &= \{U_{1,i}^n\}, \quad \boldsymbol{W}_1^n = \{W_{1,i}^n\}, \\ \boldsymbol{U}_2^n &= \{U_{2,i}^n\}, \quad \boldsymbol{W}_2^n = \{W_{2,i}^n\}, \\ &\{\boldsymbol{F}_1(\boldsymbol{U}_1^{n+1}, \boldsymbol{U}_2^{n+1})\}_i = F_1(U_{1,i}^{n+1}, U_{2,i}^{n+1}), \\ &\{\boldsymbol{F}_2(\boldsymbol{U}_1^{n+1}, \boldsymbol{U}_2^{n+1})\}_i = F_2(U_{1,i}^{n+1}, U_{2,i}^{n+1}). \end{aligned}$$

Hence the algebraic problem to be solved is

$$\begin{aligned} M(\boldsymbol{U}_1^{n+1} - \boldsymbol{U}_1^{n-1}) &= -2\Delta t K \boldsymbol{F}_1(\boldsymbol{U}_1^{n+1}, \boldsymbol{U}_2^{n+1}) - \gamma \Delta t K M^{-1} K(\boldsymbol{U}_1^{n+1} + \boldsymbol{U}_1^{n-1}), \\ M(\boldsymbol{U}_2^{n+1} - \boldsymbol{U}_2^{n-1}) &= -2\Delta t K \boldsymbol{F}_2(\boldsymbol{U}_1^{n+1}, \boldsymbol{U}_2^{n+1}) - \gamma \Delta t K M^{-1} K(\boldsymbol{U}_2^{n+1} + \boldsymbol{U}_2^{n-1}). \end{aligned}$$

This system is equivalent to

$$(M + \gamma\Delta t K M^{-1} K + \Delta t K L^1)\boldsymbol{U}_1^{n+1} = M\boldsymbol{U}_1^{n-1} - \gamma\Delta t K M^{-1} K \boldsymbol{U}_1^{n-1} - \Delta t K \boldsymbol{b}^1,$$
$$(M + \gamma\Delta t K M^{-1} K + \Delta t K L^2)\boldsymbol{U}_2^{n+1} = M\boldsymbol{U}_2^{n-1} - \gamma\Delta t K M^{-1} K \boldsymbol{U}_2^{n-1} - \Delta t K \boldsymbol{b}^2,$$

where L^j is a diagonal $J \times J$ matrix and

$$L^1_{i,i} = (U^n_{1,i})^2 + 2D(U^n_{2,i} + 1)^2,$$
$$L^2_{i,i} = (U^n_{2,i})^2 + 2D(U^n_{1,i} + 1)^2,$$
$$\{\boldsymbol{b}^1\}_i = (U^n_{1,i})^2 U^{n-1}_{1,i} - 2U^n_{1,i} + 2D(U^{n-1}_{1,i} + 2)(U^n_{2,i} + 1)^2,$$
$$\{\boldsymbol{b}^2\}_i = (U^n_{2,i})^2 U^{n-1}_{2,i} - 2U^n_{2,i} + 2D(U^{n-1}_{2,i} + 2)(U^n_{1,i} + 1)^2.$$

Note that $M + \gamma\Delta t K M^{-1} K + K L^j$ is a banded symmetric positive definite matrix, hence for given $\boldsymbol{U}_1^0$, $\boldsymbol{U}_2^0$, $\boldsymbol{U}_1^1$, $\boldsymbol{U}_2^1$ we can solve this explicitly; for the one dimensional case we solve the systems using Cholesky decomposition.

5.2 Linear Stability Analysis

5.2.1 One Dimensional Case

We consider a coupled pair of Cahn-Hilliard Equations:
Find $\{u_1(x,t), u_2(x,t)\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\frac{\partial u_1}{\partial t} = \Delta w_1 \quad \text{in } \Omega, t > 0, \tag{5.2.1a}$$
$$\frac{\partial u_2}{\partial t} = \Delta w_2 \quad \text{in } \Omega, t > 0, \tag{5.2.1b}$$
$$w_1 = -\gamma\Delta u_1 + \phi(u_1) + 2D\Psi_1(u_1, u_2), \tag{5.2.1c}$$
$$w_2 = -\gamma\Delta u_2 + \phi(u_2) + 2D\Psi_2(u_1, u_2), \tag{5.2.1d}$$

where

$$\psi(r) = \frac{1}{4}(r^2-1)^2, \qquad \phi(r) = \psi'(r), \tag{5.2.1e}$$

$$\Psi_1(r,s) = (r+1)(s+1)^2, \tag{5.2.1f}$$

$$\Psi_2(r,s) = (s+1)(r+1)^2, \tag{5.2.1g}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{5.2.1h}$$

$$u_1(x,0) = u_1^0(x), \quad u_2(x,0) = u_2^0(x) \quad \text{on } \Omega. \tag{5.2.1i}$$

For the one-dimensional problem with $\Omega = (0,1)$, we assume the solution of the linearised problem is of the form

$$u_1(x,t) = m_1 + \xi_n^1 \cos(n\pi x) F_1(t), \tag{5.2.2a}$$

$$u_2(x,t) = m_2 + \xi_n^2 \cos(n\pi x) F_2(t). \tag{5.2.2b}$$

The linear Taylor expansion of ψ' about $u_i(x,t) = m_i$ for $i = 1,2$, is given by

$$\psi'(u_i(x,t)) \approx \psi'(m_i) + \psi''(m_i)(u_i - m_i).$$

Noting

$$\psi'(r) = r(r^2-1), \qquad \psi''(r) = 3r^2 - 1,$$

we have

$$\begin{aligned}\psi'(u_1(x,t)) &\approx m_1^3 - m_1 + (3m_1^2 - 1)(u_1 - m_1),\\ &= m_1^3 - m_1 + (3m_1^2 - 1)\xi_n^1 \cos(n\pi x) F_1(t),\end{aligned} \tag{5.2.3a}$$

and

$$\begin{aligned}\psi'(u_2(x,t)) &\approx m_2^3 - m_2 + (3m_2^2 - 1)(u_2 - m_2),\\ &= m_2^3 - m_2 + (3m_2^2 - 1)\xi_n^2 \cos(n\pi x) F_2(t).\end{aligned} \tag{5.2.3b}$$

The expansion of Ψ_i for $i = 1, 2$, about $u_1(x,t) = m_1$ and $u_2(x,t) = m_2$ is given by

$$\begin{aligned}\Psi_i(u_1(x,t), u_2(x,t)) \approx \Psi_i(m_1, m_2) &+ \Psi_{i,u_1}(m_1, m_2)(u_1 - m_1)\\ &+ \Psi_{i,u_2}(m_1, m_2)(u_2 - m_2).\end{aligned}$$

Noting

$$\begin{aligned}\Psi_1(r,s) &= \Psi(r,s) = (r+1)(s+1)^2,\\ \frac{\partial}{\partial r}\Psi_1(r,s) &= \Psi_r(r,s) = (s+1)^2,\\ \frac{\partial}{\partial s}\Psi_1(r,s) &= \Psi_s(r,s) = 2(r+1)(s+1),\\ \Psi_2(r,s) &= \Psi(s,r) = (s+1)(r+1)^2,\\ \frac{\partial}{\partial r}\Psi_2(r,s) &= \Psi_r(s,r) = 2(r+1)(s+1),\\ \frac{\partial}{\partial s}\Psi_2(r,s) &= \Psi_s(s,r) = (s+1)^2,\end{aligned}$$

we have

$$\begin{aligned}\Psi_1(u_1(x,t), u_2(x,t)) &\approx (m_1+1)(m_2+1)^2 + (m_2+1)^2(u_1 - m_1)\\ &\quad + 2(m_1+1)(m_2+1)(u_2 - m_2),\\ &= (m_1+1)(m_2+1)^2 + (m_2+1)^2\xi_n^1\cos(n\pi x)F_1(t)\\ &\quad + 2(m_1+1)(m_2+1)\xi_n^2\cos(n\pi x)F_2(t), \qquad (5.2.4a)\end{aligned}$$

and

$$\begin{aligned}\Psi_2(u_1(x,t), u_2(x,t)) &\approx (m_2+1)(m_1+1)^2 + (m_1+1)^2(u_1 - m_1)\\ &\quad + 2(m_1+1)(m_2+1)(u_2 - m_2),\\ &= (m_2+1)(m_1+1)^2 + (m_1+1)^2\xi_n^2\cos(n\pi x)F_2(t)\\ &\quad + 2(m_1+1)(m_2+1)\xi_n^1\cos(n\pi x)F_1(t). \qquad (5.2.4b)\end{aligned}$$

On substituting (5.2.1c) into (5.2.1a) and (5.2.1d) into (5.2.1b), noting (5.2.3a–

5.2.4b) and simplifying, we see that the problem reduces to a linear system of ordinary differential equations given by

$$\begin{aligned}\xi_n^1 \frac{dF_1(t)}{dt} = &-\gamma(n\pi)^4 \xi_n^1 F_1(t) - \Big((3m_1^2-1)(n\pi)^2 + 2D(m_2+1)^2(n\pi)^2\Big)\xi_n^1 F_1(t) \\ &- 4D(m_1+1)(m_2+1)(n\pi)^2 \xi_n^2 F_2(t),\end{aligned} \tag{5.2.5a}$$

and

$$\begin{aligned}\xi_n^2 \frac{dF_2(t)}{dt} = &-\gamma(n\pi)^4 \xi_n^2 F_2(t) - \Big((3m_2^2-1)(n\pi)^2 + 2D(m_1+1)^2(n\pi)^2\Big)\xi_n^2 F_2(t) \\ &- 4D(m_1+1)(m_2+1)(n\pi)^2 \xi_n^1 F_1(t).\end{aligned} \tag{5.2.5b}$$

Here we have noted for $i = 1, 2$, that

$$\begin{aligned}\Delta(-\gamma\Delta u_i) &= -\gamma(n\pi)^4 \xi_n^i \cos(n\pi x) F_i(t), \\ \frac{du_i(t)}{dt} &= \xi_n^i \cos(n\pi x)\frac{dF_i(t)}{dt}, \\ \Delta\phi(u_i) &= (1-3m_i^2)\xi_n^i (n\pi)^2 \cos(n\pi x) F_i(t),\end{aligned}$$

and

$$\begin{aligned}\Delta\Psi_1(u_1,u_2) = &-(m_2+1)^2 \xi_n^1 (n\pi)^2 \cos(n\pi x) F_1(t) \\ &- 2(m_1+1)(m_2+1)\xi_n^2 (n\pi)^2 \cos(n\pi x) F_2(t), \\ \Delta\Psi_2(u_1,u_2) = &-(m_1+1)^2 \xi_n^2 (n\pi)^2 \cos(n\pi x) F_2(t) \\ &- 2(m_1+1)(m_2+1)\xi_n^1 (n\pi)^2 \cos(n\pi x) F_1(t).\end{aligned}$$

In terms of vectors we can express (5.2.5a–b) as

$$\frac{d\boldsymbol{F}(t)}{dt} = (-\gamma n^4 \pi^4 I - n^2\pi^2 A)\boldsymbol{F}(t), \tag{5.2.6}$$

where

$$\boldsymbol{F}(t) = [\xi_n^1 F_1(t),\ \xi_n^2 F_2(t)]^T, \tag{5.2.7}$$

$$A = (a_{ij}) \quad \text{for} \quad i,j = 1,2, \tag{5.2.8}$$

$$a_{11} = (3m_1^2 - 1) + 2D(m_2+1)^2,$$

$$a_{21} = 4D(m_1+1)(m_2+1),$$

$$a_{12} = 4D(m_1+1)(m_2+1),$$

$$a_{22} = (3m_2^2 - 1) + 2D(m_1+1)^2.$$

Thus the solution of (5.2.6) is given by

$$\boldsymbol{F}(t) = \exp\big((-\gamma n^4\pi^4 I - n^2\pi^2 A)\,t\big)\boldsymbol{F}(0). \tag{5.2.9}$$

To see interesting behaviour, i.e. growth of one or more of the component u_1 and u_2, as t increases, we need at least one of the eigenvalues of A to be smaller than $-\gamma n^2\pi^2 < 0$. A simple calculation reveals that the eigenvalues of the matrix A are

$$\lambda_1 = \frac{3}{2}m_1^2 + \frac{3}{2}m_2^2 + D(m_1^2 + m_2^2 + 2m_1 + 2m_2 + 2) - 1 + \frac{1}{2}\sqrt{Q(m_1,m_2)}, \tag{5.2.10a}$$

$$\lambda_2 = \frac{3}{2}m_1^2 + \frac{3}{2}m_2^2 + D(m_1^2 + m_2^2 + 2m_1 + 2m_2 + 2) - 1 - \frac{1}{2}\sqrt{Q(m_1,m_2)}, \tag{5.2.10b}$$

where

$$\begin{aligned} Q(m_1,m_2) = {} & D^2\big(4m_1^4 + 16m_1^3 + 56m_1^2m_2^2 + 112m_1^2m_2 + 80m_1^2 + 112m_1m_2^2 \\ & + 224m_1m_2 + 128m_1 + 4m_2^4 + 16m_2^3 + 80m_2^2 + 128m_2 + 64\big) \\ & + D\big(- 12m_1^4 - 24m_1^3 + 24m_1^2m_2^2 + 24m_1^2m_2 + 24m_1m_2^2 \\ & - 12m_2^4 - 24m_2^3\big) + 9m_1^4 - 18m_1^2m_2^2 + 9m_2^4. \end{aligned} \tag{5.2.10c}$$

Since A is symmetric we have real eigenvalues, i.e. $Q(m_1,m_2) \geq 0$. By noting the symmetry of the eigenvalues it is obvious that for $D \geq 0.25$, we have $\lambda_1 \geq 0$.

Thus we need only seek where λ_2 is negative and less than $-\gamma n^2\pi^2$. To see where $\lambda_2 < 0$, for $D = 0.5$, we plot the curve of the projection onto the m_1–m_2 plane of the intersection of the surface $\lambda_2(m_1, m_2)$ with the plane m_1–m_2, which is equivalent to plotting $Det(A) = \lambda_1 \times \lambda_2 = 0$. The result is shown in Figure 5.1.

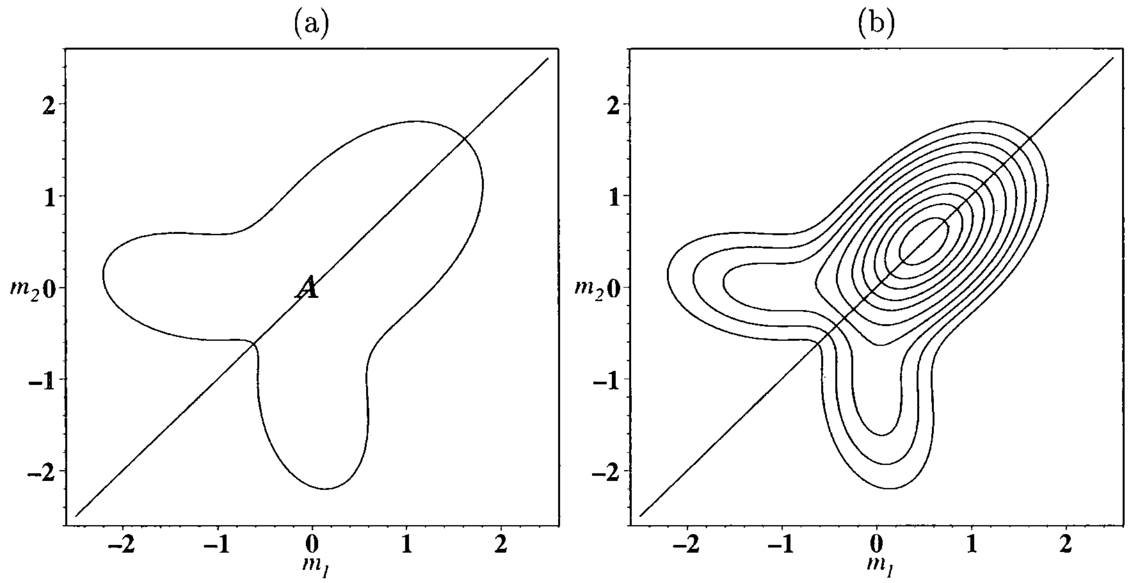

Figure 5.1: (a) The region $\boldsymbol{A}$ indicates where $\lambda_2 < 0$, i.e. where we expect growth to occur; (b) The contour plot of region $\boldsymbol{A}$ and its boundary.

For the case $m_1 = m_2 =: m$, the eigenvalues of A are

$$\begin{aligned} \lambda_1 &= (3+6D)m^2 + 12Dm + (6D-1), && (5.2.11a) \\ \lambda_2 &= (3-2D)m^2 - 4Dm - (2D+1). && (5.2.11b) \end{aligned}$$

Setting $\lambda_i = 0,\ i = 1, 2$, and solving for m we obtain

$$m = \begin{cases} \dfrac{-12D \pm \sqrt{12 - 48D}}{6 + 12D} & \text{by setting } \lambda_1 = 0, \\[2ex] \dfrac{4D \pm \sqrt{12 + 16D}}{6 - 4D} & \text{by setting } \lambda_2 = 0. \end{cases} \tag{5.2.12}$$

This shows that for $D \geq 0.25$, $\lambda_1 \geq 0$. Thus for the case we consider, i.e. $D = 0.5$, the growth may occur if $2m^2 - 2m - 2 \leq -\gamma n^2\pi^2$ or

$$m \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{5 - 2\gamma n^2\pi^2}\,,\ \frac{1}{2} + \frac{1}{2}\sqrt{5 - 2\gamma n^2\pi^2}\right). \tag{5.2.13}$$

For $n = 1$, from (5.2.2a–b), we have

$$u_1(x,t) = m + \cos(\pi x)\xi_1^1 F_1(t), \tag{5.2.14a}$$

$$u_2(x,t) = m + \cos(\pi x)\xi_1^2 F_2(t). \tag{5.2.14b}$$

Using the formula given by Bernstein and So in [7], we can write

$$\exp\left([-\gamma\pi^4 I - \pi^2 A]t\right) = \begin{pmatrix} \frac{1}{2}(\exp(c_1 t) + \exp(c_2 t)) & \frac{1}{2}(\exp(c_1 t) - \exp(c_2 t)) \\ \frac{1}{2}(\exp(c_1 t) - \exp(c_2 t)) & \frac{1}{2}(\exp(c_1 t) + \exp(c_2 t)) \end{pmatrix}, \tag{5.2.15}$$

where $c_i = -\gamma\pi^4 - \pi^2\lambda_i$, $i = 1,2$. Hence noting (5.2.9) and (5.2.15), we can rewrite (5.2.14a–b) as

$$\begin{aligned} u_1(x,t) = m + \frac{1}{2}\cos(\pi x)\big(&F_1(0)[\exp(c_1 t) + \exp(c_2 t)] \\ &+ F_2(0)[\exp(c_1 t) - \exp(c_2 t)]\big), \end{aligned} \tag{5.2.16a}$$

$$\begin{aligned} u_2(x,t) = m + \frac{1}{2}\cos(\pi x)\big(&F_1(0)[\exp(c_1 t) - \exp(c_2 t)] \\ &+ F_2(0)[\exp(c_1 t) + \exp(c_2 t)]\big). \end{aligned} \tag{5.2.16b}$$

5.3 Numerical Simulations

5.3.1 One Dimensional Case

Numerical simulations in one space dimension were performed with $\Omega = (0,1)$. In all simulations we take $N = 101$, $\gamma = 0.0025$, $\boldsymbol{Z}_1^0 = \boldsymbol{Z}_2^0 = 0.51$ and $D = 0.5$.

To choose μ we ran one experiment with $\mu \in (0,2)$. We varied the initial conditions and Δt, and set $T = 10$, $TOL = 1 \times 10^{-10}$. The value that required on average fewer iterations, as seen in Table 5.1, is $\mu = 0.1$. We used this value in all our simulations. We cannot say this value is the best for all the cases since μ depends on N.

Given initial guesses U_1^0 and U_2^0, to solve (5.1.11a-b) for each node the Newton

U_1^0	U_2^0	μ	average iteration	min	max	Δt
0.00	0.00	0.10	15.081	13	178	0.001
0.25	0.50	0.40	4.197	2	19	0.001
0.25	-0.50	0.10	6.246	1	284	0.001
0.75	0.75	0.20	5.286	5	15	0.001
-0.75	-0.75	0.60	1.241	1	9	0.001
-0.50	0.50	0.10	4.849	1	81	0.001
0.50	0.50	0.10	8.792	8	49	0.001
-0.50	-0.50	0.10	20.616	8	104	0.001
0.50	-0.50	0.10	4.757	1	83	0.001
0.00	0.00	0.10	18.118	14	178	0.0005
0.00	0.00	0.10	20.134	4	178	0.00025

Table 5.1: Different initial guesses to choose better μ.

method was used. We set its initial guess based on the value of the right hand side of (5.1.11a-b), say y. If $y \geq 0$, we set the initial guess for the Newton method $y+0.1$, otherwise $y - 0.1$. Using this value as an initial guess, we found that for each node the Newton method never failed to converge to the solution of (5.1.11a-b). We used $TOL \times 10^{-1}$ as a tolerance to stop the iteration in the Newton method.

To find the unique solution of the systems (5.1.12a-b) we used Cholesky decomposition, i.e. exact for the tolerance provided. We moved to the next time level if $||U_i^{n,j+1}(x_i) - U_i^{n,j+\frac{1}{2}}(x_i)||_\infty < TOL$.

Note that in all simulations the iterative method we used gave solutions that conserved mass.

A comparison

We consider the problem (5.2.1a–h) with the following initial conditions

$$u_1(x,0) = u_1^0(x) = \zeta_1 \cos(\pi x), \tag{5.3.1a}$$

$$u_2(x,0) = u_2^0(x) = \zeta_2 \cos(\pi x), \tag{5.3.1b}$$

where ζ_1, ζ_2 are small.

Comparing to (5.2.2a–b), and setting $m_1 = m_2 = 0$, $\xi_1^1 = \zeta_1$ and $\xi_1^2 = \zeta_2$ we have

$$F_1(0) = 1, \text{ and } F_2(0) = 1.$$

Hence from (5.2.16a–b) the solution of the linearised problem is

$$u_1(x,t) = \frac{1}{2}\cos(\pi x)\left([\zeta_1 + \zeta_2]\exp(c_1 t) + [\zeta_1 - \zeta_2]\exp(c_2 t)\right), \tag{5.3.2a}$$

$$u_2(x,t) = \frac{1}{2}\cos(\pi x)\left([\zeta_1 + \zeta_2]\exp(c_1 t) + [\zeta_2 - \zeta_1]\exp(c_2 t)\right), \tag{5.3.2b}$$

where $c_1 = -\gamma\pi^4 - 2\pi^2$ and $c_2 = -\gamma\pi^4 + 2\pi^2$. If $\zeta_1 = \zeta_2$, we have

$$u_i(x,t) = \zeta_1 \cos(\pi x)\exp(c_1 t),$$

and it follows that u_i, $i = 1, 2$, decays to zero as t increases. If $\zeta_1 \neq \zeta_2$, the solution will grow as t increases provided that $\gamma < 2/\pi^2$. For example, for $\zeta_1 = \zeta_2/2$, we have

$$u_1(x,t) = \frac{1}{2}\zeta_1\cos(\pi x)\left(3\exp(c_1 t) - \exp(c_2 t)\right),$$
$$u_2(x,t) = \frac{1}{2}\zeta_1\cos(\pi x)\left(3\exp(c_1 t) + \exp(c_2 t)\right),$$

while for $\zeta_1 = 2\zeta_2$, we have

$$u_1(x,t) = \frac{1}{4}\zeta_1\cos(\pi x)\left(3\exp(c_1 t) + \exp(c_2 t)\right),$$
$$u_2(x,t) = \frac{1}{4}\zeta_1\cos(\pi x)\left(3\exp(c_1 t) - \exp(c_2 t)\right).$$

For the first example, we performed a simulation where $\zeta_1 = \zeta_2 = 0.001$, with $TOL = 1 \times 10^{-9}$, $\lambda_1 = 2.0$ and $\lambda_2 = -2.0$. Here we see that the maximum errors (see Figure 5.2.(a)) decrease linearly as the time-steps are halved, which is what we expect from the error analysis. The solutions are also in agreement with what we expect, i.e. u_i, $i = 1, 2$, decays to zero as t increases (see Figure 5.2.(b)). We also did simulations with $TOL = 1 \times 10^{-10}$. The results are the same.

In the second example, simulations were performed by setting $\zeta_2 = 0.001$, $\zeta_1 = 2\zeta_2$, and $\zeta_2 = 0.002$, $\zeta_1 = \zeta_2/2$ respectively with $TOL = 1 \times 10^{-9}$. Their maximum errors are depicted in Figures 5.3.(a) and 5.4.(a) respectively. We notice that the

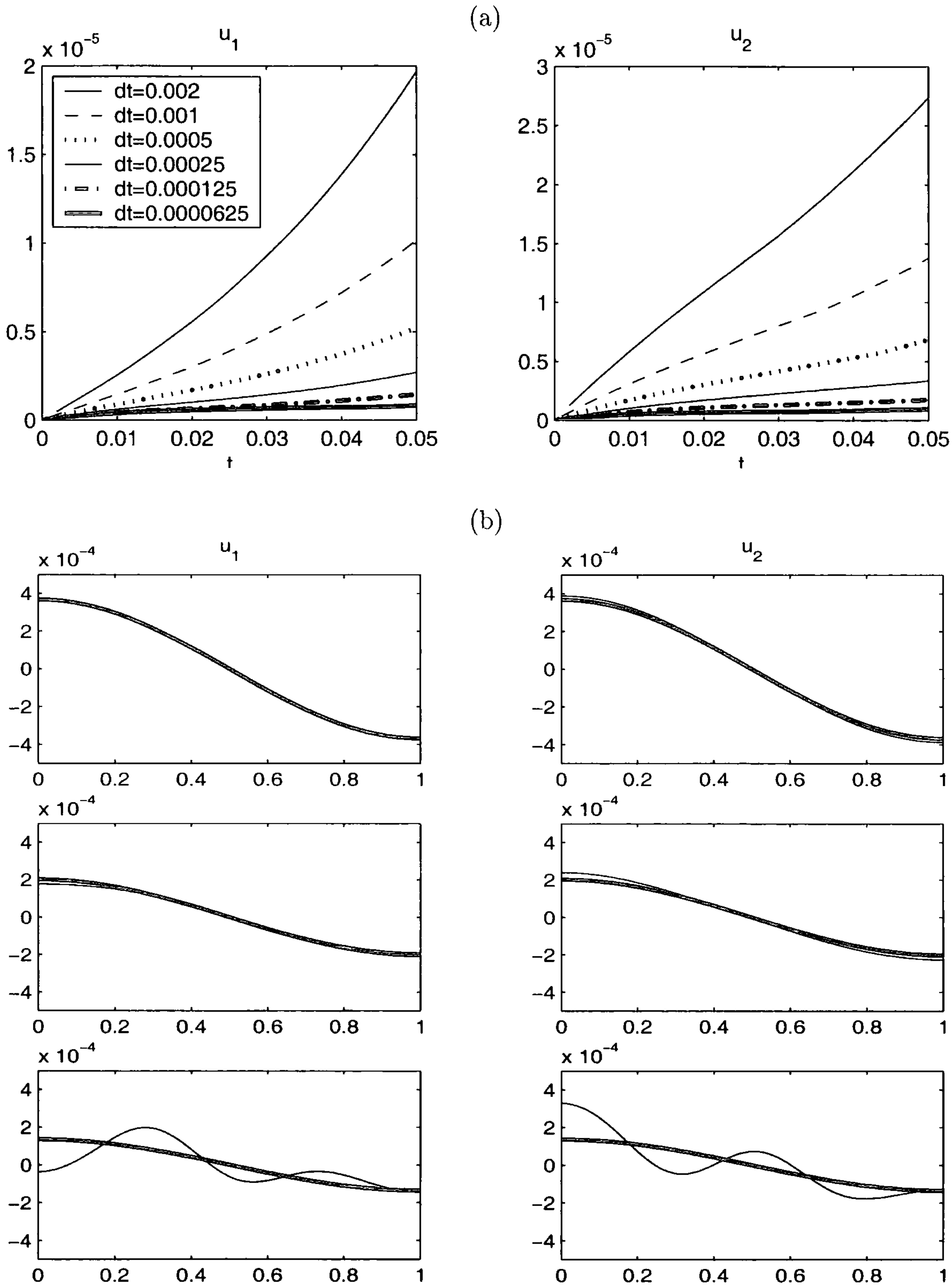

Figure 5.2: (a) Maximum Errors of u_i, $i = 1, 2$, for $\zeta_1 = \zeta_2 = 0.001$ with $u_1^0 = \zeta_1 \cos(\pi x)$ and $u_2^0 = \zeta_2 \cos(\pi x)$; (b) Plot of the numerical solution and the linear stability analysis solution of $u_i, i = 1, 2$, for $\zeta_1 = \zeta_2 = 0.001$ at $t = 0.05, 0.08, 0.1$.

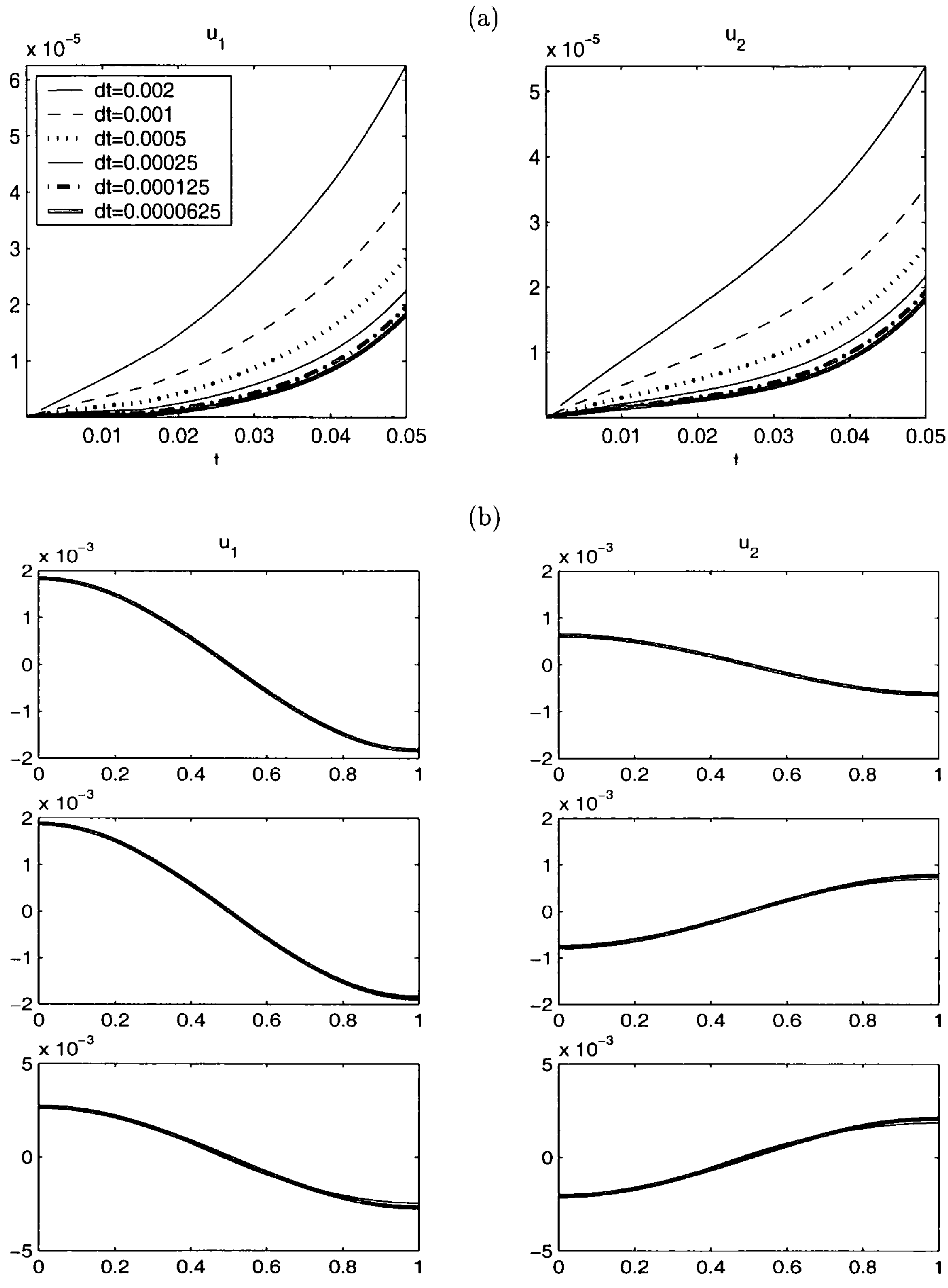

Figure 5.3: (a) Maximum Errors of u_i, $i = 1, 2$, for $\zeta_1 = 0.002$, $\zeta_2 = 0.001$, with $u_1^0 = \zeta_1 \cos(\pi x)$ and $u_2^0 = \zeta_2 \cos(\pi x)$; (b) Plot of the numerical solution and the linear stability analysis solution of $u_i, i = 1, 2$, for $\zeta_1 = 0.002$, $\zeta_2 = 0.001$ at $t = 0.01, 0.05, 0.08$.

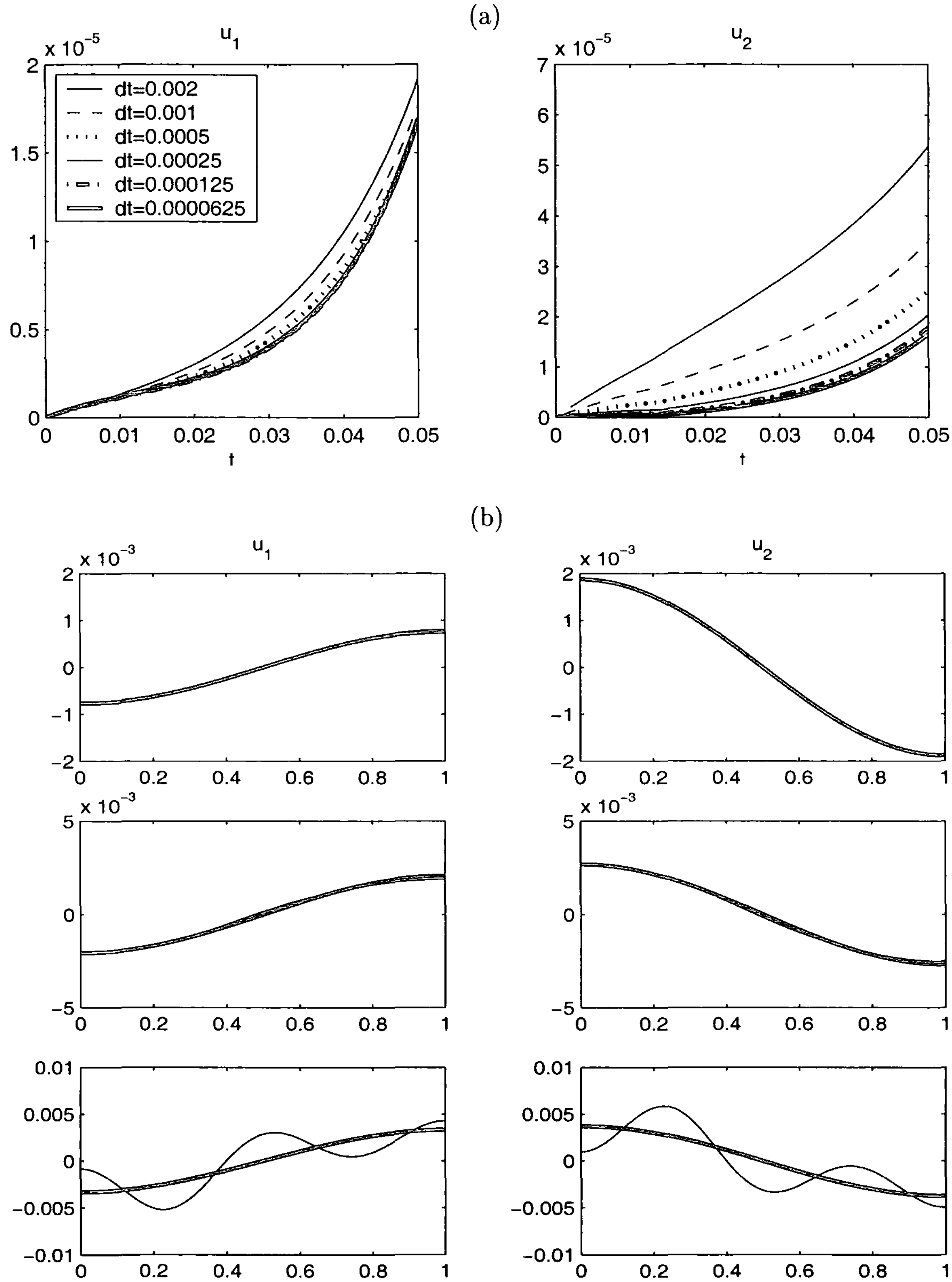

Figure 5.4: (a) Maximum Errors of u_i, $i = 1, 2$, for $\zeta_1 = 0.001$, $\zeta_2 = 0.002$, with $u_1^0 = \zeta_1 \cos(\pi x)$ and $u_2^0 = \zeta_2 \cos(\pi x)$; (b) Plot of the numerical solution and the linear stability analysis solution of $u_i, i = 1, 2$, for $\zeta_1 = 0.001$, $\zeta_2 = 0.002$ at $t = 0.05, 0.08, 0.1$.

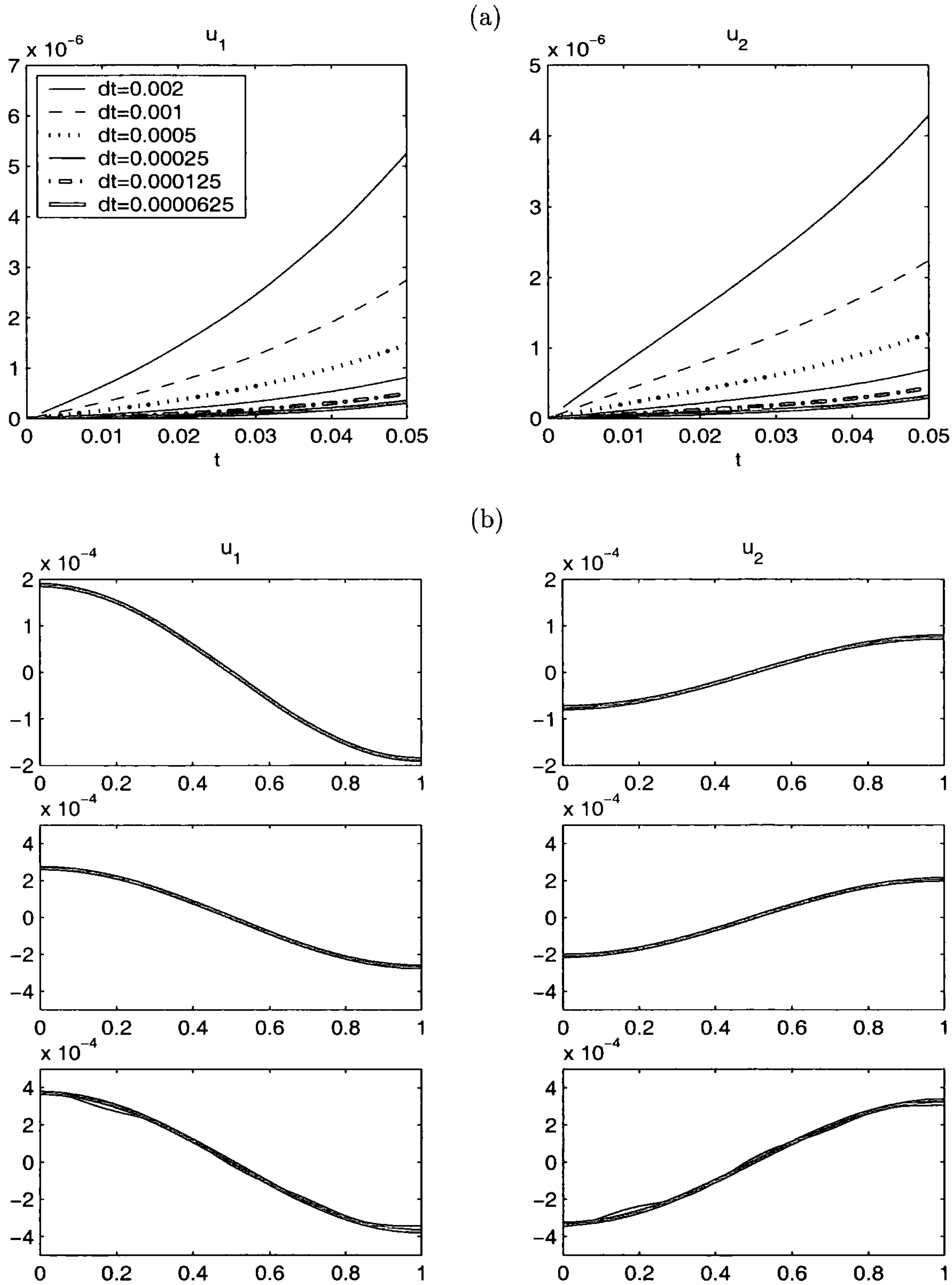

Figure 5.5: (a) Maximum Errors of u_i, $i = 1, 2$, for $\zeta_1 = 0.0002$, $\zeta_2 = 0.0001$, with $u_1^0 = \zeta_1 \cos(\pi x)$ and $u_2^0 = \zeta_2 \cos(\pi x)$; (b) Plot of the numerical solution and the linear stability analysis solution of $u_i, i = 1, 2$, for $\zeta_1 = 0.0002$, $\zeta_2 = 0.0001$ at $t = 0.05, 0.08, 0.1$.

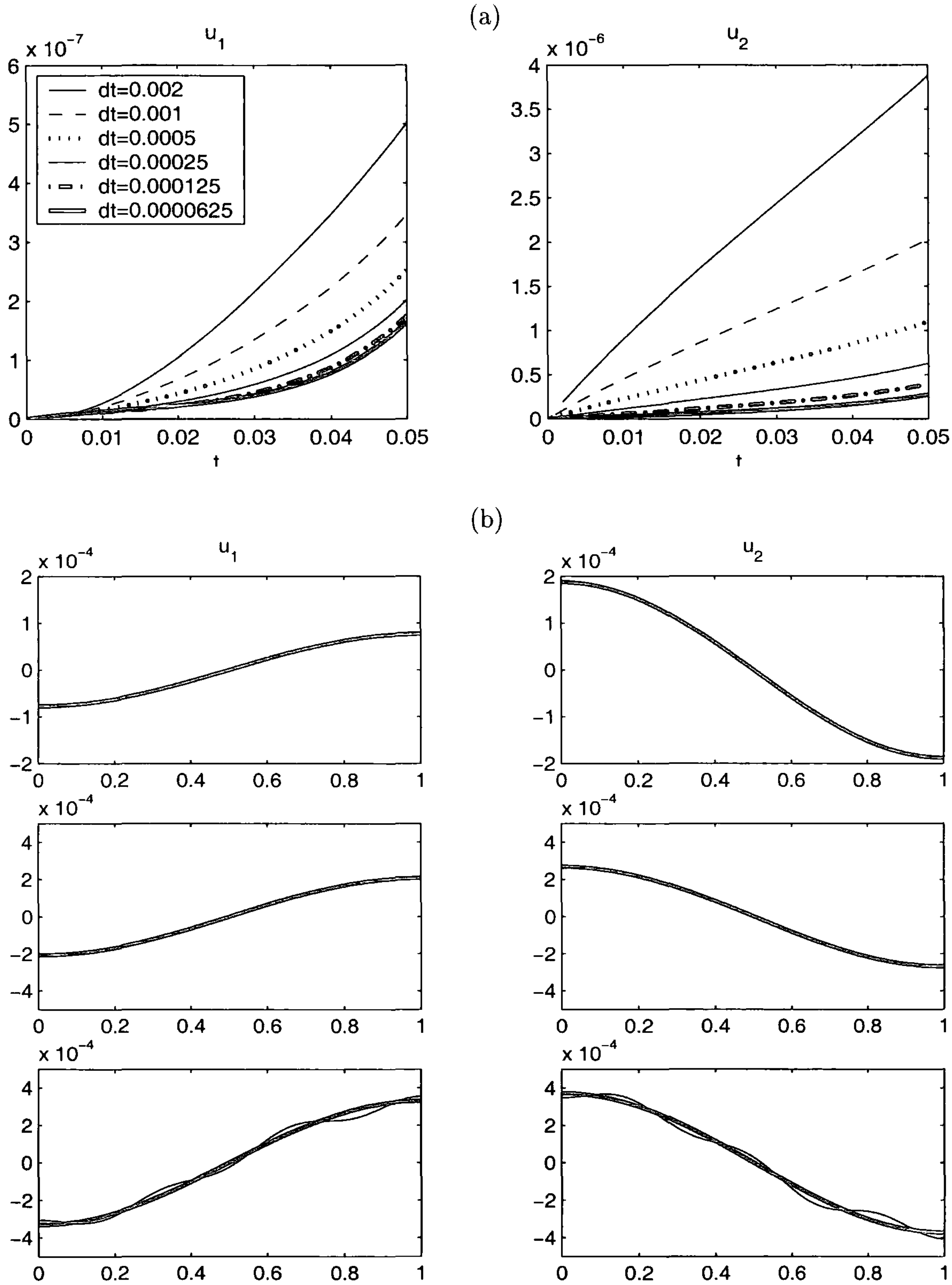

Figure 5.6: (a) Maximum Errors of u_i, $i = 1,2$, for $\zeta_1 = 0.0001$, $\zeta_2 = 0.0002$, with $u_1^0 = \zeta_1 \cos(\pi x)$ and $u_2^0 = \zeta_2 \cos(\pi x)$; (b) Plot of the numerical solution and the linear stability analysis solution of $u_i, i = 1,2$, for $\zeta_1 = 0.0001$, $\zeta_2 = 0.0002$ at $t = 0.05, 0.08, 0.1$.

errors do not decrease linearly as the time-steps are halved. However the behaviours of the approximations are quite good in following the linear stability analysis solution as can be seen in Figures 5.3.(b) and 5.4.(b) respectively. This ensures that our approximations are good enough.

We also did several simulations by taking $\zeta_2 = 0.0001$, $\zeta_1 = 2\zeta_2$ and $\zeta_2 = 0.0002$, $\zeta_1 = \zeta_2/2$. In the first run we took $TOL = 1 \times 10^{-9}$. We could not see the linear convergence rate of the maximum errors using these parameters. Then we reduced the tolerance of the method by a factor of 10 until we could see the linear convergence rate of the maximum errors. The maximum errors for $\zeta_2 = 0.0001$, $\zeta_1 = 2\zeta_2$, and $\zeta_2 = 0.0002$, $\zeta_1 = \zeta_2/2$ respectively with $TOL = 1 \times 10^{-12}$ are depicted respectively in Figures 5.5.(a) and 5.6.(a) . We notice that the maximum errors of the approximations are almost linear as the time-steps are halved for $\zeta_2 = 0.0001$ and $\zeta_1 = 2\zeta_2$. However this is not the case for $\zeta_2 = 0.0002$ and $\zeta_1 = \zeta_2/2$. We believe this is because the scheme is not symmetric. The behaviour of these solutions can be seen in Figures 5.5.(b) and 5.6.(b) respectively. These figures are what we expect, i.e. u_i, $i = 1, 2$, grow as t increases.

Simulations with no exact solutions

For simulations in this section we use as initial conditions

$$U_i^0 = U_i^m + \varsigma(x), \tag{5.3.3}$$

where $\varsigma(x)$ is a random perturbation of the state $U = 0$ with values distributed uniformly between -0.05 and $+0.05$. In all simulations we set $TOL = 1 \times 10^{-10}$ and $\Delta t = 0.001$. We ran several simulations with different initial guesses, although some of them, which are located in the top right hand quadrant and its border, do not have physical meaning according to the model as mention in Chapter 1. Our aim is to see whether the results are in agreement with the stability region we obtained in Section 5.2. For each figure, the last plot indicates that the final numerical solution plotted is stationary, that is U_i^n does not change from one time level to the next.

Figure 5.8.(a)–(c) show the simulations with initial guesses $\widehat{A}, \widehat{C}$ and $\widehat{F}$ respec-

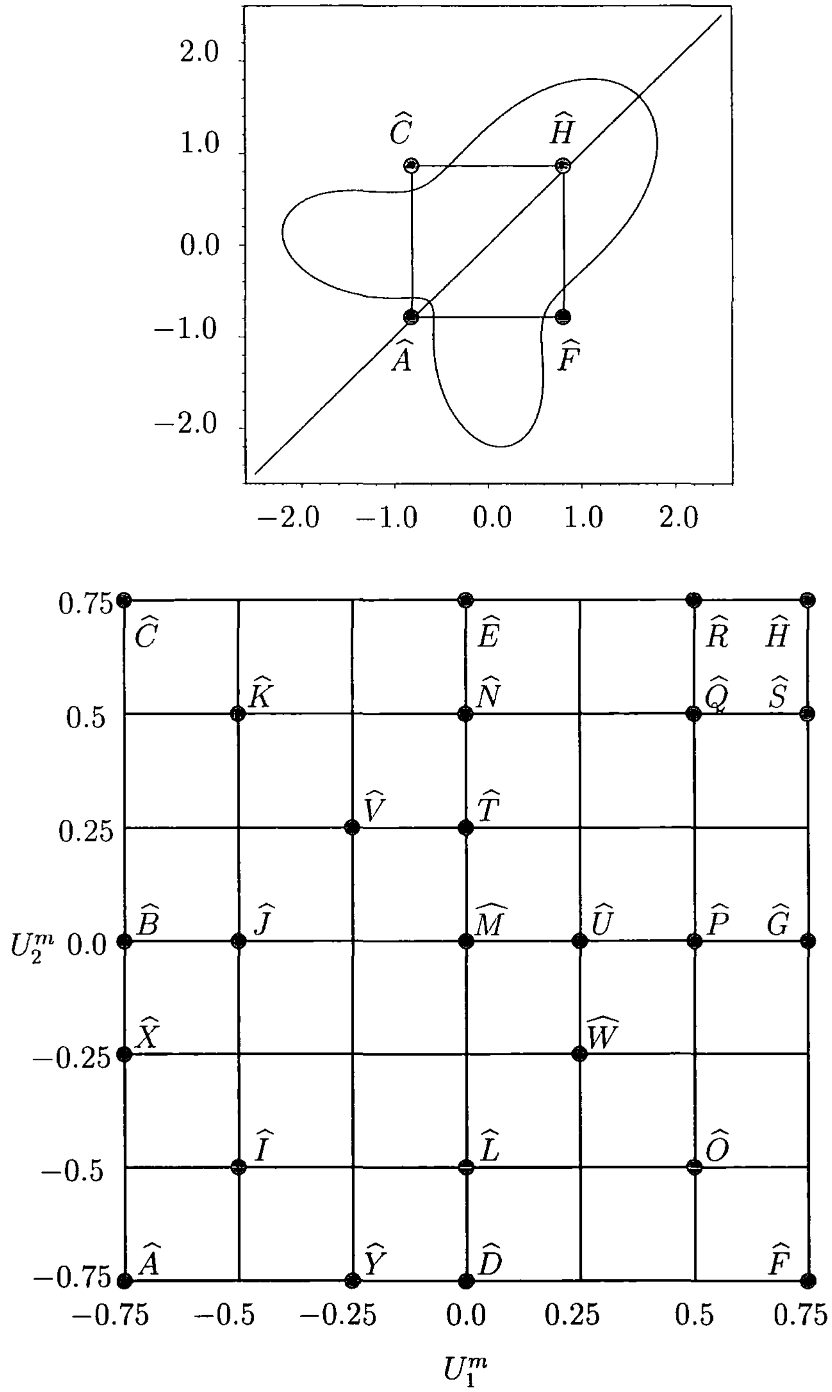

Figure 5.7: The symbol ⊛ indicates the values of U_i^m in the initial guesses for the simulations we performed in this section.

tively. As can be seen in Figure 5.7, these initial guesses are located outside the region $\boldsymbol{A}$ of the stability region. Here we have stationary solutions, which match the stability region condition (see Figure 5.1).

The rest of the initial guesses are located inside the A-area of the stability region. After computing each of their eigenvalues, using (5.2.10a–b), we can compute the values of $c_i = -\gamma\pi^4 - \pi^2\lambda_i$ (see Table 5.2). Using these c_i in the solutions (5.2.16a–b), and plotting $(F_1(0) + F_2(0))\exp(c_1 t) + (F_1(0) - F_2(0))\exp(c_2 t)$ and $(F_1(0) + F_2(0))\exp(c_1 t) + (F_2(0) - F_1(0))\exp(c_2 t)$ we can judge the behaviour of the solutions at the early stage of the given initial guesses.

Initial	U_1^0	U_2^0	c_1	c_2
$\widehat{X}$	-0.75	-0.25	-13.25166595	7.829818293
$\widehat{Y}$	-0.25	-0.75	-13.25166595	7.829818293
$\widehat{I}$	-0.50	-0.50	-5.178324930	4.691279474
$\widehat{J}$	-0.50	0.00	-12.58052823	12.09348277
$\widehat{K}$	-0.50	0.50	-27.90580951	7.679555244
$\widehat{V}$	-0.25	0.25	-21.86310670	16.44125904
$\widehat{L}$	0.00	-0.50	-12.58052823	12.09348277
$\widehat{O}$	-0.50	-0.50	-27.90580951	7.679555244
$\widehat{W}$	0.25	-0.25	-21.86310670	16.44125904
$\widehat{M}$	0.00	0.00	-19.98273154	19.49568608
$\widehat{N}$	0.00	0.50	-39.82457085	19.59831658
$\widehat{P}$	0.50	0.00	-39.82457085	19.59831658
$\widehat{Q}$	0.50	0.50	-64.39595163	24.43048828
$\widehat{R}$	0.50	0.75	-80.43773010	23.20045932
$\widehat{S}$	0.75	0.50	-80.43773010	23.20045932

Table 5.2: The value of c_i for different initial guesses.

Using the analysis above, we expect using the initial conditions $\widehat{X}$, $\widehat{Y}$ and $\widehat{I}$, at the early stage, that the solution will go to a steady state before growing. $\widehat{I}$ will grow more slowly than $\widehat{X}$ and $\widehat{Y}$. This observation is in agreement with our computational results (see Figure 5.9.a,b,c, for $\widehat{X}$, $\widehat{Y}$ and $\widehat{I}$ respectively). We also expect that at the early stage the solutions using initial guesses $\widehat{K}$ and $\widehat{O}$ will grow more quickly than those with $\widehat{J}$ and $\widehat{L}$, but slower than those with $\widehat{V}$ and $\widehat{W}$. See Figures 5.10 and 5.11 for the matching behaviour in our simulations.

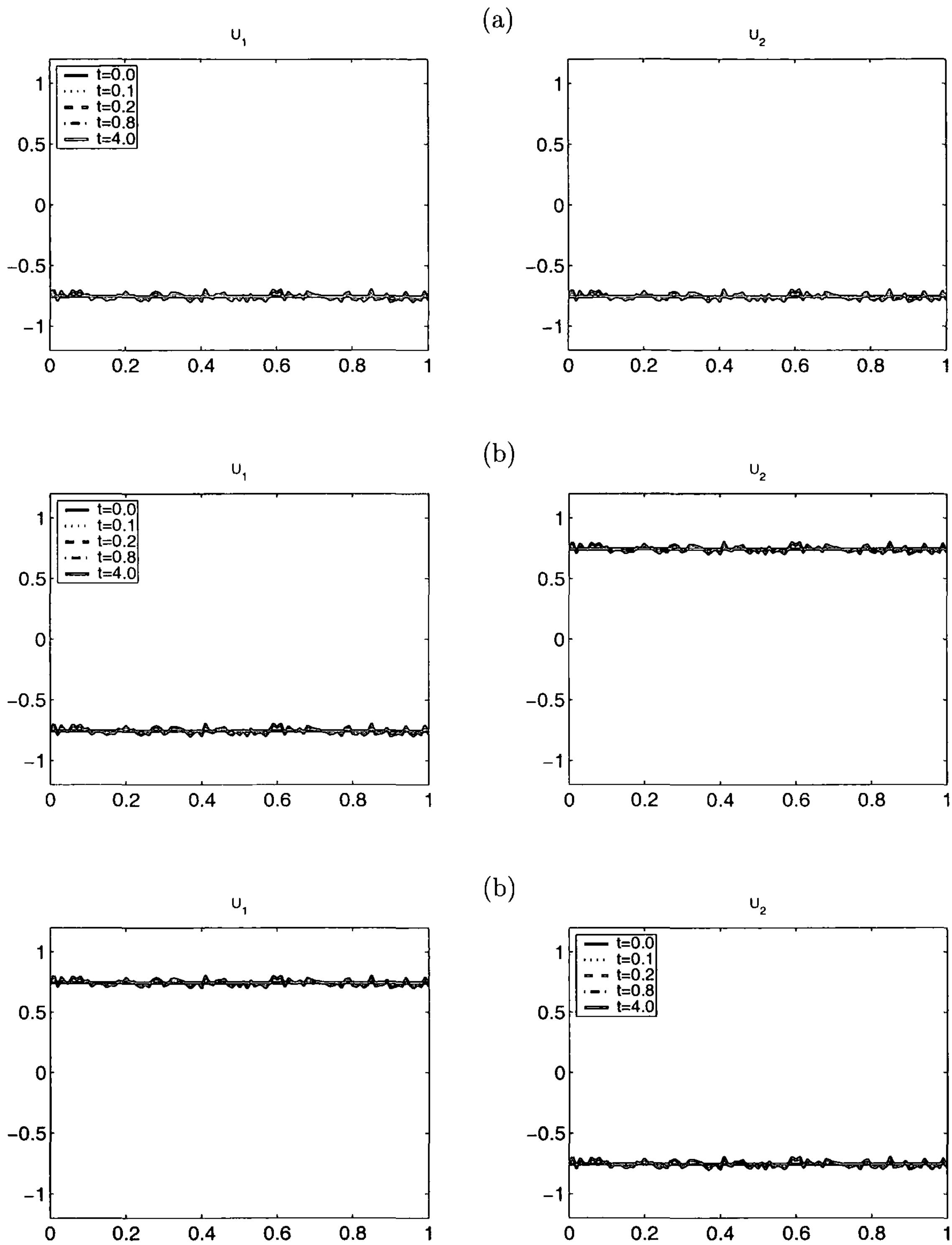

Figure 5.8: The evolution from the initial conditions that are random perturbations of the uniform state: (a) $U_1 = -0.75$ and $U_2 = -0.75$ at $t = 0.0$, 0.1, 0.2, 0.8, 4.0; (b) $U_1 = -0.75$ and $U_2 = 0.75$ at $t = 0.0$, 0.1, 0.2, 0.8, 4.0; (c) $U_1 = 0.75$ and $U_2 = -0.75$ at $t = 0.0$, 0.1, 0.2, 0.8, 4.0.

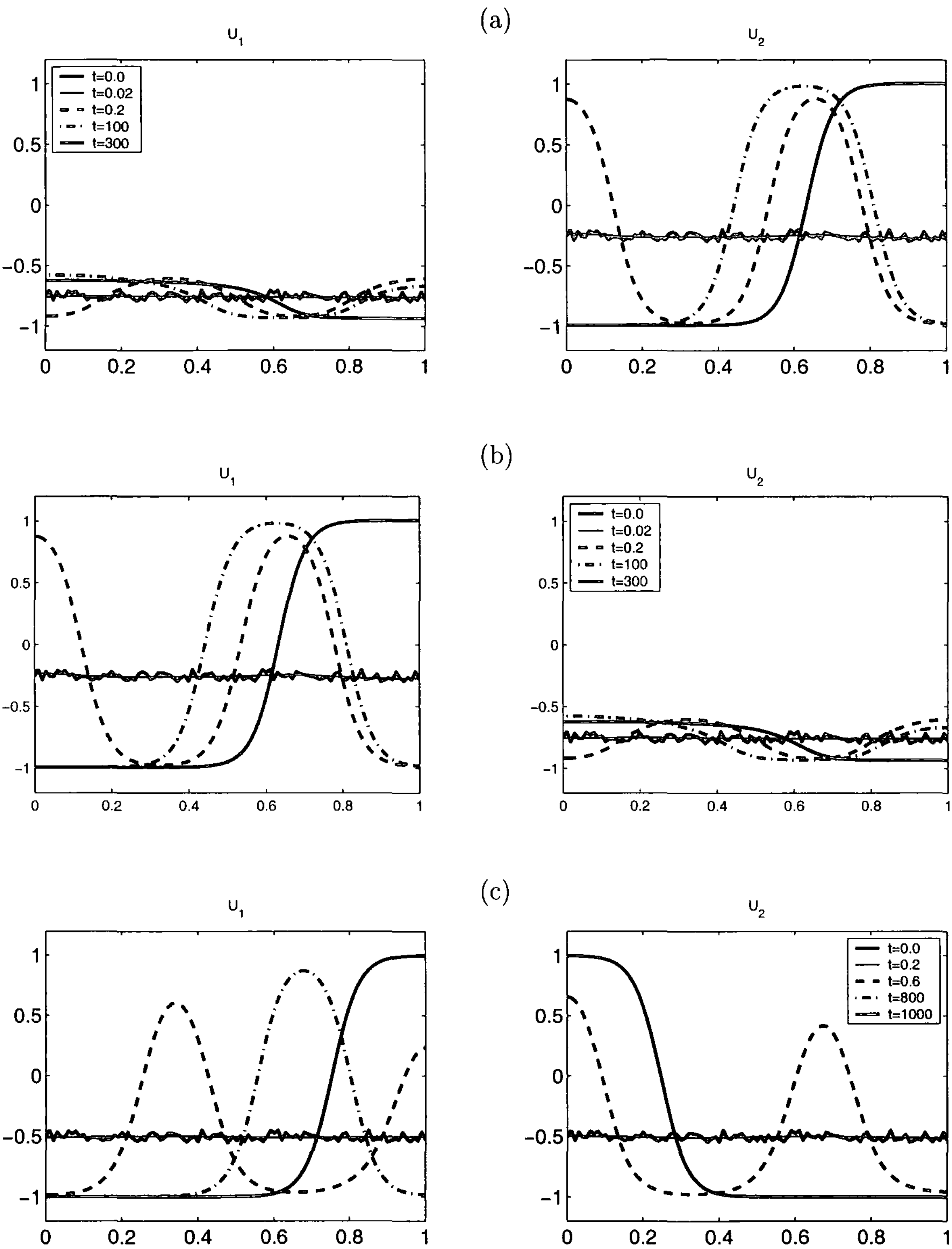

Figure 5.9: The evolution from the initial conditions that are random perturbations of the uniform state: (a) $U_1 = -0.75$ and $U_2 = -0.25$ at $t = 0.0$, 0.08, 0.2, 100, 300; (b) $U_1 = -0.25$ and $U_2 = -0.75$ at $t = 0.0$, 0.08, 0.2, 100, 300; (c) $U_1 = -0.5$ and $U_2 = -0.5$ at $t = 0.0$, 0.4, 0.8, 800, 1000.

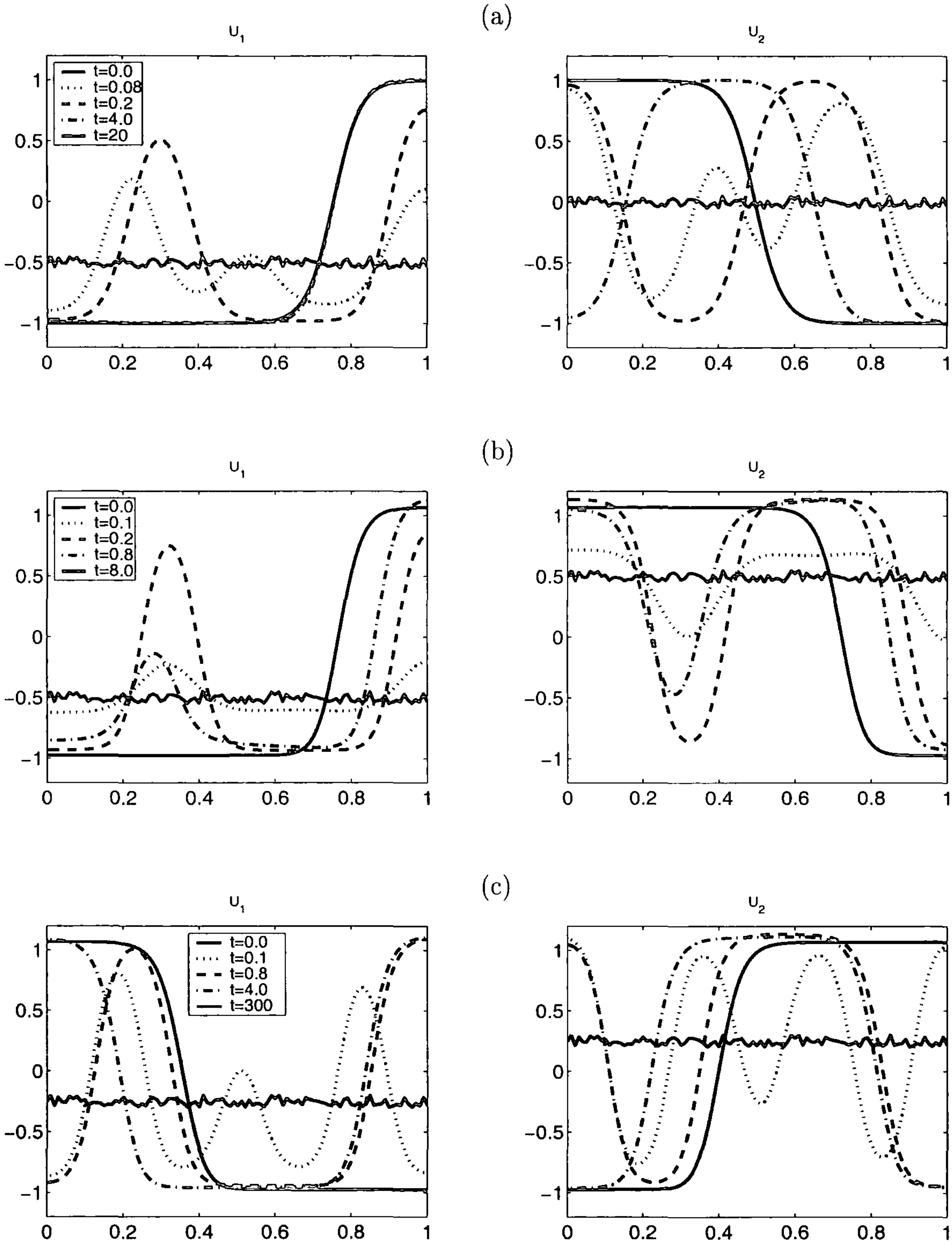

Figure 5.10: The evolution from the initial conditions that are random perturbations of the uniform state: (a) $U_1 = -0.5$ and $U_2 = 0.0$ at $t = 0.0$, 0.08, 0.2, 4.0, 20; (b) $U_1 = -0.5$ and $U_2 = 0.5$ at $t = 0.0$, 0.1, 0.2, 0.8, 8.0; (c) $U_1 = -0.25$ and $U_2 = 0.25$ at $t = 0.0$, 0.1, 0.8, 4.0, 300.

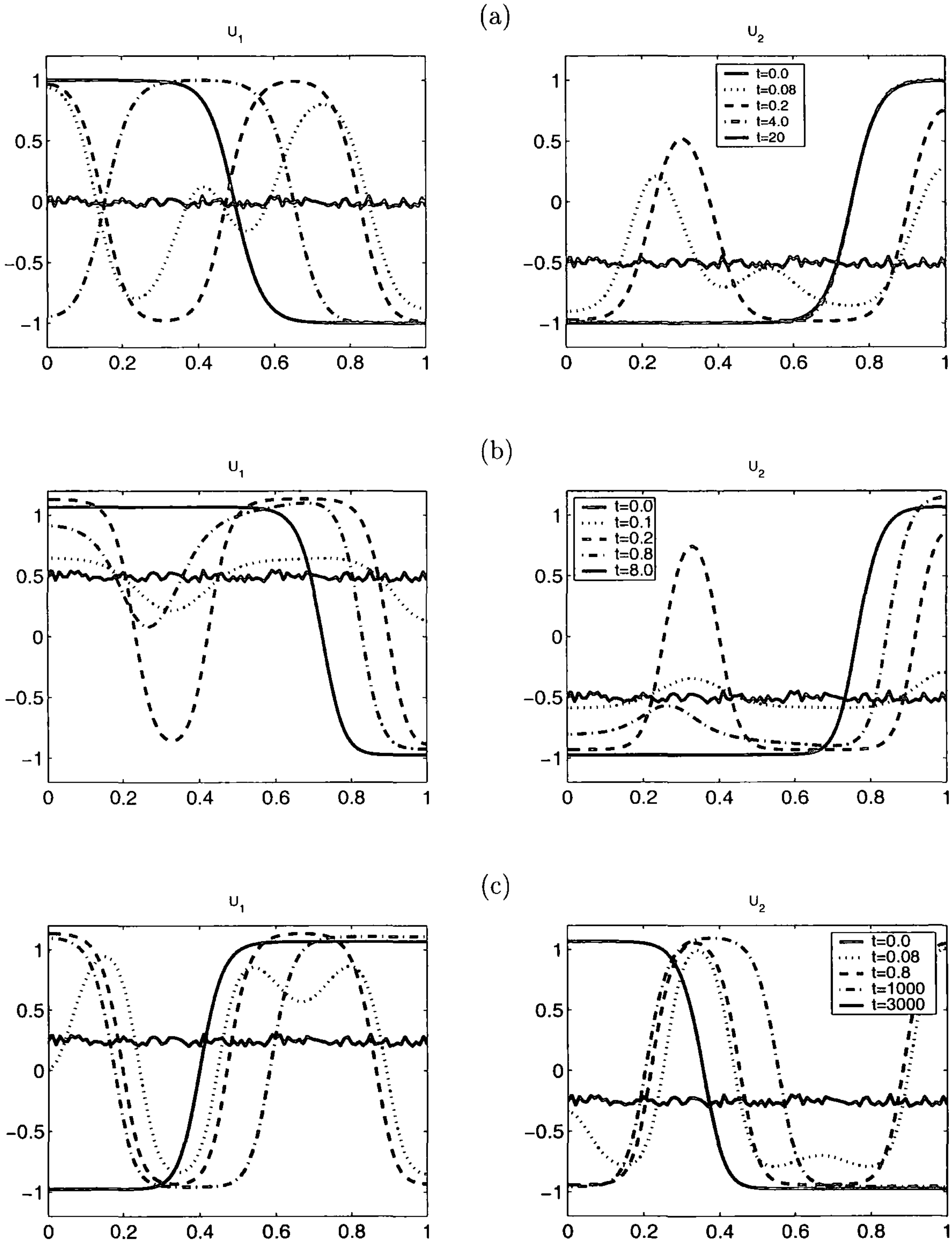

Figure 5.11: The evolution from the initial conditions that are random perturbations of the uniform state: (a) $U_1 = 0.0$ and $U_2 = -0.5$ at $t = 0.0, 0.08, 0.2, 4.0, 20$; (b) $U_1 = 0.5$ and $U_2 = -0.5$ at $t = 0.0, 0.1, 0.2, 0.8, 8.0$; (c) $U_1 = 0.25$ and $U_2 = -0.25$ at $t = 0.0, 0.08, 0.8, 1000, 3000$.

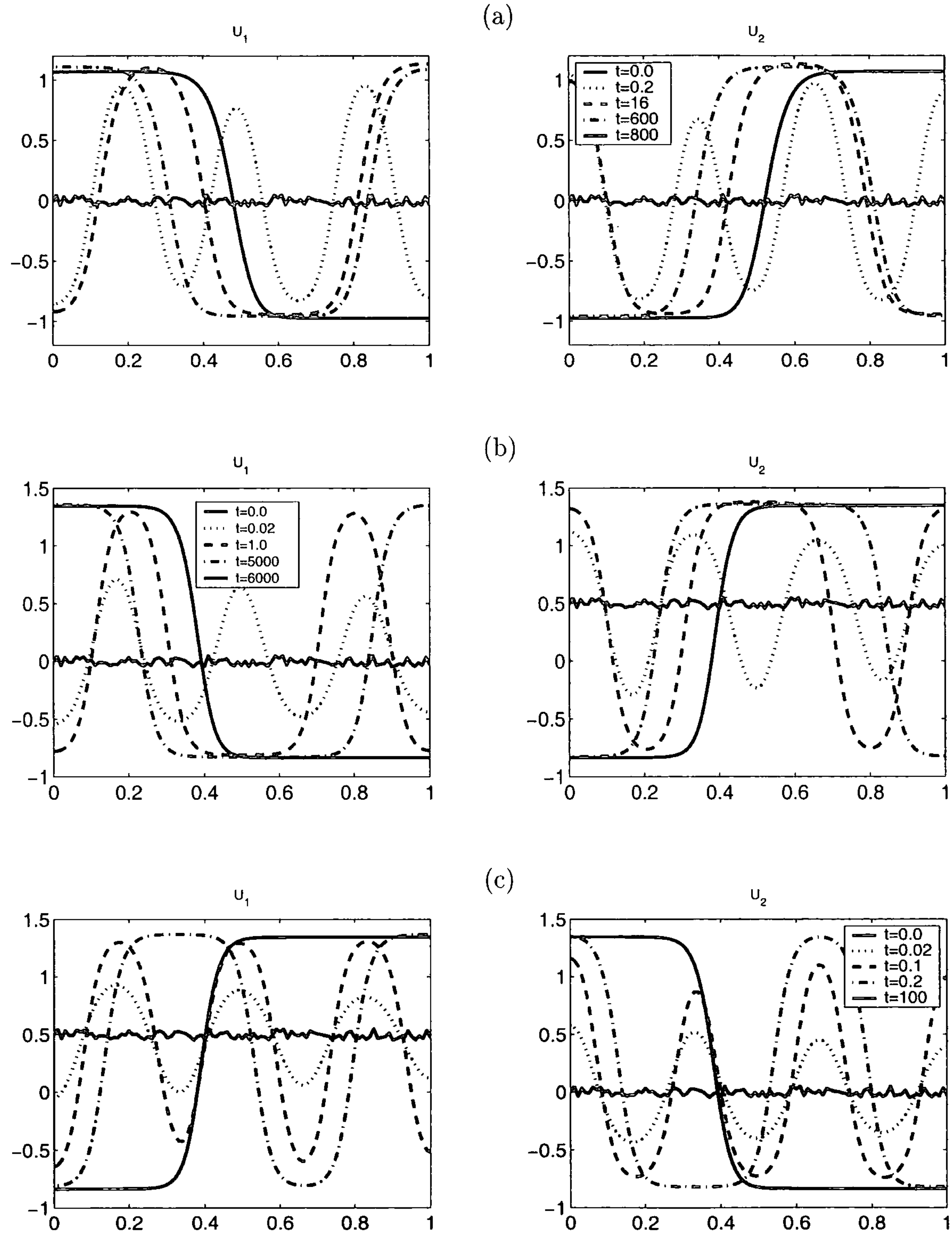

Figure 5.12: The evolution from the initial conditions that are random perturbations of the uniform state: (a) $U_1 = 0.0$ and $U_2 = 0.0$ at $t = 0.0$, 0.2, 16, 600, 800; (b) $U_1 = 0.0$ and $U_2 = 0.5$ at $t = 0.0$, 0.02, 1.0, 5000, 6000; (c) $U_1 = 0.5$ and $U_2 = 0.0$ at $t = 0.0$, 0.02, 0.1, 0.2, 100.

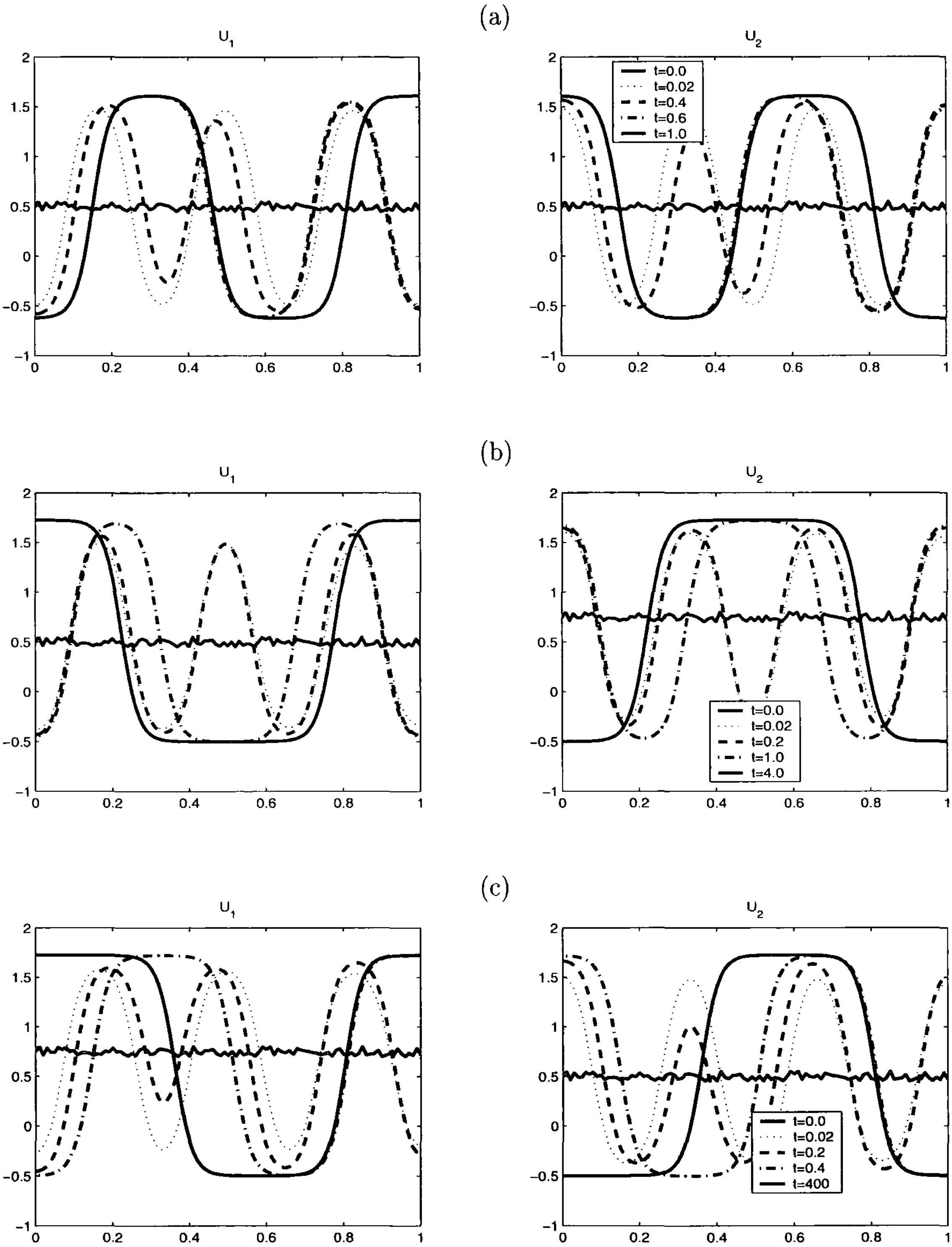

Figure 5.13: The evolution from the initial conditions that are random perturbations of the uniform state: (a) $U_1 = 0.5$ and $U_2 = 0.5$ at $t = 0.0$, 0.02, 0.4, 0.6, 1.0; (b) $U_1 = 0.5$ and $U_2 = 0.75$ at $t = 0.0$, 0.02, 0.2, 1.0, 4.0; (c) $U_1 = 0.75$ and $U_2 = 0.5$ at $t = 0.0$, 0.02, 0.2, 0.4, 400.

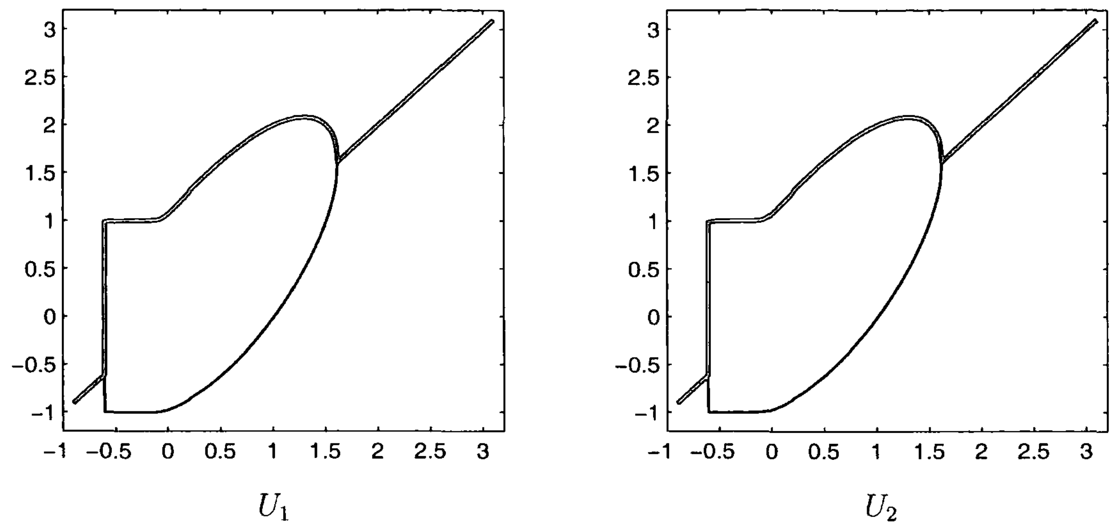

Figure 5.14: Maximum and minimum U_1 and U_2 at $t = 1000$.

In all simulations, including those using initial conditions in the first quadrant $\widehat{E}$, $\widehat{G}$, $\widehat{H}$, $\widehat{Q}$, $\widehat{R}$, $\widehat{S}$, $\widehat{T}$, $\widehat{U}$, the results are consistent with the corresponding stability region (see Figure 5.1), where for all initial guesses in region $\boldsymbol{A}$ we get growth of the solutions.

We notice that in the Figures 5.8–5.13 the growth of the approximate solutions, which are initially much less than 1, may grow to be close to 2 (see the y-axis of Figures 5.12 and 5.13). Do the solutions blow-up for increasing time t? To tell us what happens to the solutions for increasing time t, we ran a simulation with the initial guesses $U_1^0 = U_2^0 \in (-1, 3.2)$. Again our simulations violated the physical meaning of the model. We plotted the maximum and minimum values of U_1, U_2, for $t = 1000$, as can be seen in Figure 5.14. The Figures are consistent with the stability region (see Figure 5.1) and the interval given in (5.2.13), where we have stationary solutions outside of this interval.

We note something mathematically interesting in Figure 5.14. The maximum and minimum values of ± 1 change as we enter the domain having no physical meaning, i.e. the top right hand quadrant including its boundary. A rough explanation can be stated as follows: Consider minimising the free energy (2.2.14) with mass constraints. We ignore the γ–terms. Thus we have a problem

$$\min \int_\Omega \Big(\frac{1}{4}(u_1^2 - 1)^2 + \frac{1}{4}(u_2^2 - 1)^2 + \frac{1}{2}(u_1 + 1)^2(u_2 + 1)^2\Big)dx,$$

such that

$$\int_\Omega u_1(x)dx = m_1, \qquad \int_\Omega u_2(x)dx = m_2.$$

For certain pairs of m_1 and m_2 the solution of this problem is

$$u_1 = \begin{cases} 1 & \text{in } \Omega^1_+, \\ -1 & \text{in } \Omega^1_-, \end{cases} \quad \text{and} \quad u_2 = \begin{cases} 1 & \text{in } \Omega^2_+, \\ -1 & \text{in } \Omega^2_-, \end{cases}$$

where $\Omega^1_+ \cap \Omega^2_+ = \emptyset$, $\bar{\Omega} = \bar{\Omega}^1_+ \cup \bar{\Omega}^1_- = \bar{\Omega}^2_+ \cup \bar{\Omega}^2_-$, such that

$$|\Omega| m_1 = |\Omega^1_+| - |\Omega^1_-|, \qquad |\Omega| m_2 = |\Omega^2_+| - |\Omega^2_-|.$$

This tells us that if we have growth and the growth is in the domain described above, then the maximum/minimum will be $1/-1$. Comparing the domain to the stability region in Figure 5.1 we have an area where the maximum/minimum will be $1/-1$ as depicted in Figure 5.15. Our analysis here is in agreement with the simulations we have done (see Figure 5.9–5.13).

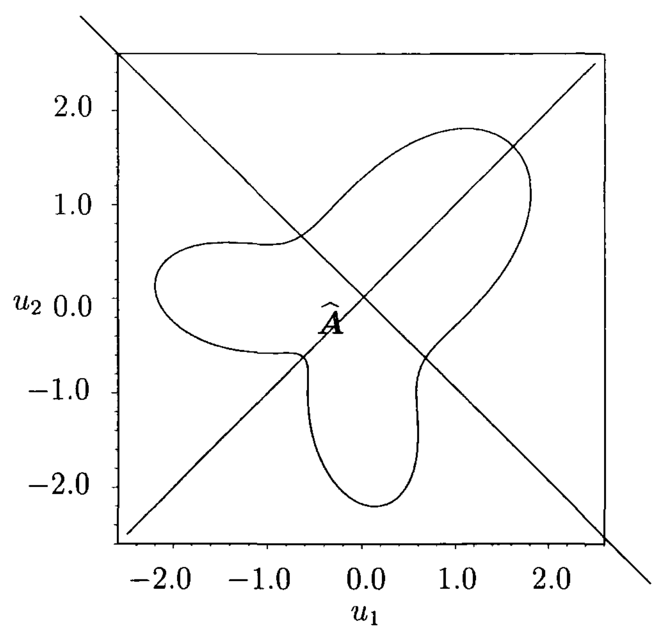

Figure 5.15: The region $\widehat{\boldsymbol{A}}$ indicated where the maximum/minimum will have values $1/-1$.

5.3.2 Two Dimensional Case

Numerical experiments in two space dimensions were performed with $\Omega = (0,1) \times (0,1)$. We took a uniform mesh consisting of a square κ of length $h = 1/64$, each of which was divided into two triangles by its north-east diagonal (see Figure 5.16). In all simulations we set $\gamma = \Delta t = 0.001$, $\boldsymbol{Z}_1^0 = \boldsymbol{Z}_2^0 = 0.51$, and $D = 0.5$.

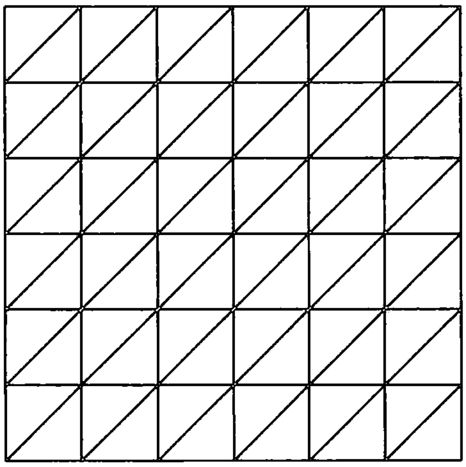

Figure 5.16: Uniform mesh

As in the one dimensional experiments we use initial guesses (5.3.3), a random perturbation of the state $U = 0$ with values distributed uniformly between -0.05 and $+0.05$. We solve (5.1.11a-b) for each node using the Newton method, applying the same technique as used for the one dimensional simulations to obtain its initial guess. The technique to stop the iteration and to move to the next time level is the same as in the one dimensional case.

To find the unique solutions of the systems (5.1.12a-b), unlike the one dimensional simulations, we used a relaxation method. We chose its 'best' parameter ω, by running a single simulation for $\omega \in (0,2)$. We picked the value which took fewest iterations on average. In this case $\omega = 1.8$.

The parameter μ was chosen by running one simulation for $\mu \in (0,4)$. Here we did not vary initial guesses or Δt as we did in the one dimensional case. We ran this simulation with the following parameters: $T = 0.01$ and $TOL = 1 \times 10^{-9}$. We chose μ so that it took fewer iteration in average, i.e. $\mu = 2$.

We did several simulations in two space dimensions, each of which conserved mass. We plotted a gray scale grid plot of U_i, that is,

$$t = 1 - (U(i,j) + U(i+1,j) + U(i,j+1) + U(i+1,j+1) + 4c)/8c, \; c = 1.2,$$

at several times. Except for Figure 5.18 the final plot of the numerical solution is stationary. The gray scale ranges from 0.1 to 0.9 with pure black/white corresponds to 0.1/0.9 representing values larger/smaller than 0.9/0.1.

The aim of our simulations is to see an agreement of the behaviour with the one dimensional case and the stability analysis at their early stages. Figure 5.17, where no growth occurs, shows the simulation with initial conditions $U_1 = U_2 = -0.75 + \varsigma(x)$. This matches the one dimension simulation (see Figure 5.8) and the stability region condition (see Figure 5.1 and 5.7).

Figures 5.18–5.20 represent respectively, the initial stage evolution from the initial guesses labelled $\widehat{Y}$, $\widehat{K}$, $\widehat{O}$ in Figure 5.7. As can be seen in the Figures, we obtain the growth as expected. In addition, analysis done as in the one dimensional simulation showed that in the early stage the solutions inherited the behaviour of the solutions in one space dimension.

Concluding Remarks

We have done some experiments incorporating a multigrid technique with the implicit algorithm to solve the linear system (5.1.12a–b). In the implementation we used a V–cycle with a Gauss-Seidel smoother, a seven point prolongation and restriction. We did not perform a prolongation and restriction while moving from grid to grid on the left hand matrix of (5.1.12a–b), instead we used the mass and stiffness matrices to construct it for each grid levels. However the result is not promising. The CPU-time required to obtain the solution using this technique was much longer than when applying the relaxation method to solve the system (5.1.12a–b). We believe that, one reason for this inefficiency is because the smallest system is of the order 25×25 which is not solvable exactly. Another reason is the dependence of our problem on the parameter μ for each grid level. The best value of μ for solving the system using a multigrid approach may be the worst for the algorithm overall (see

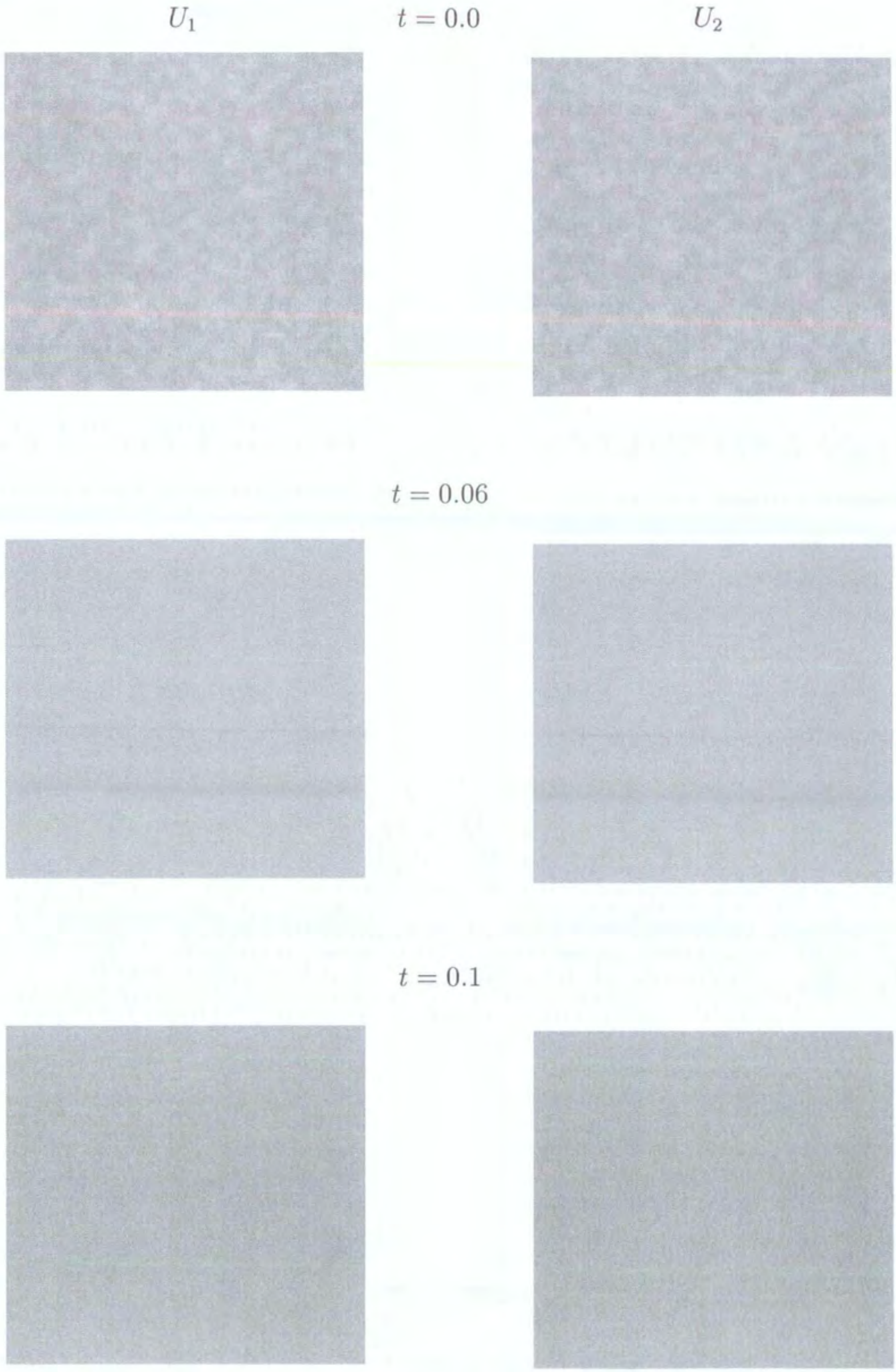

Figure 5.17: The early stage evolution from the initial conditions that are random perturbations of the uniform state $U_1 = -0.75$ and $U_2 = -0.75$ at $t = 0.0$, 0.06, 0.1.

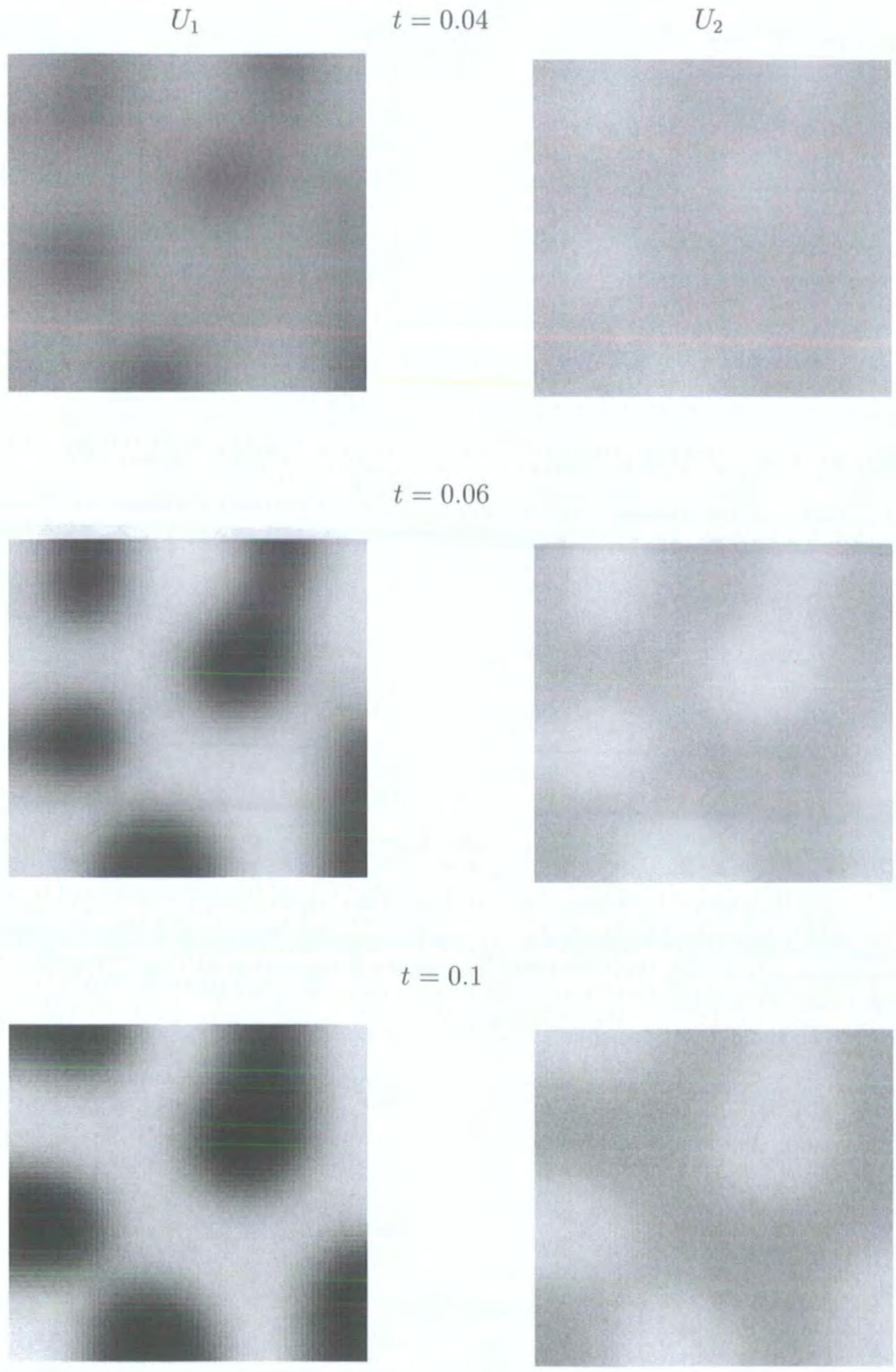

Figure 5.18: The early stage evolution from the initial conditions that are random perturbations of the uniform state $U_1 = -0.25$ and $U_2 = -0.75$ at $t = 0.04$, 0.06, 0.1.

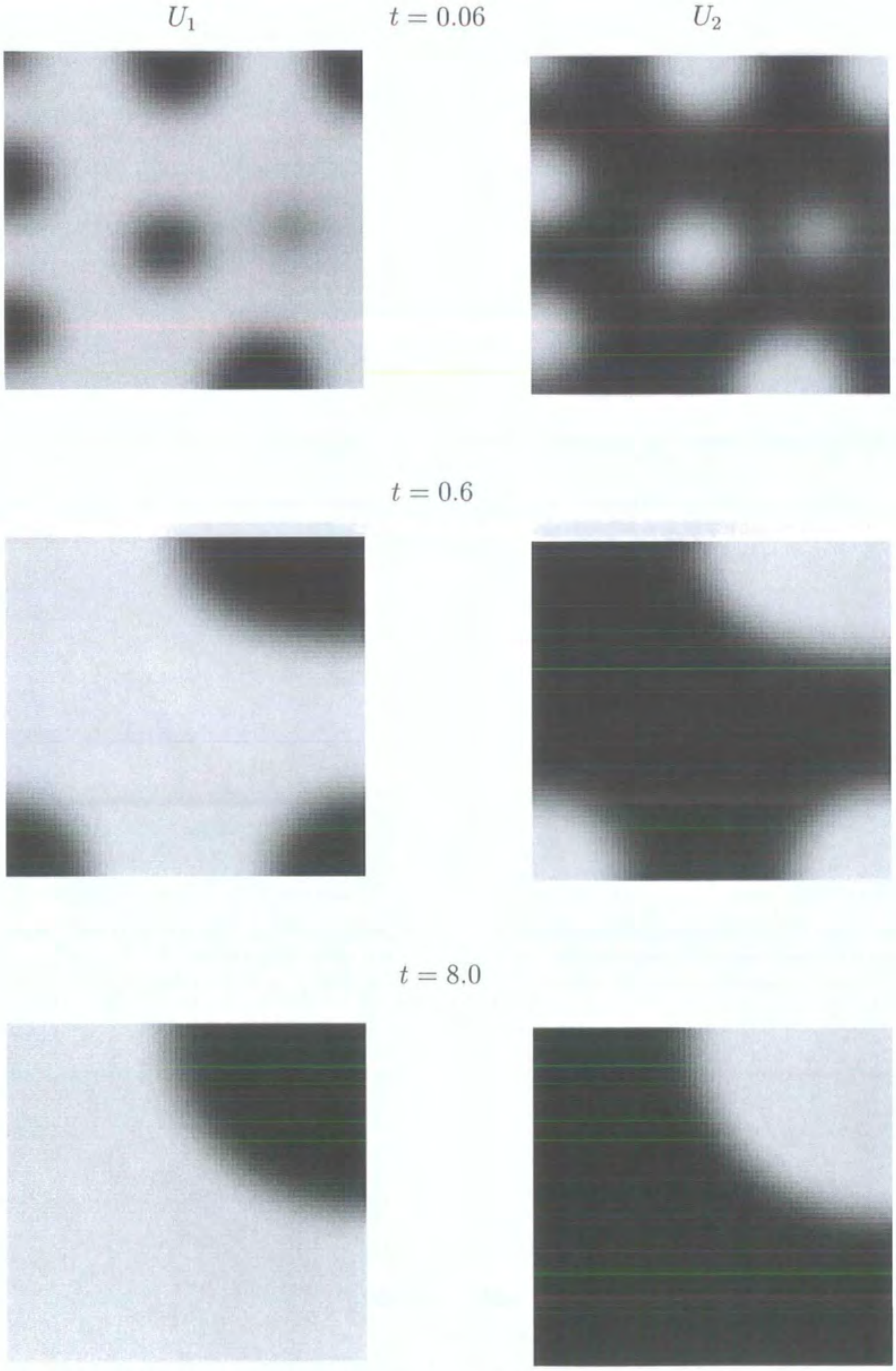

Figure 5.19: The early stage evolution from the initial conditions that are random perturbations of the uniform state $U_1 = -0.5$ and $U_2 = 0.5$ at $t = 0.06$, 0.6, 8.0.

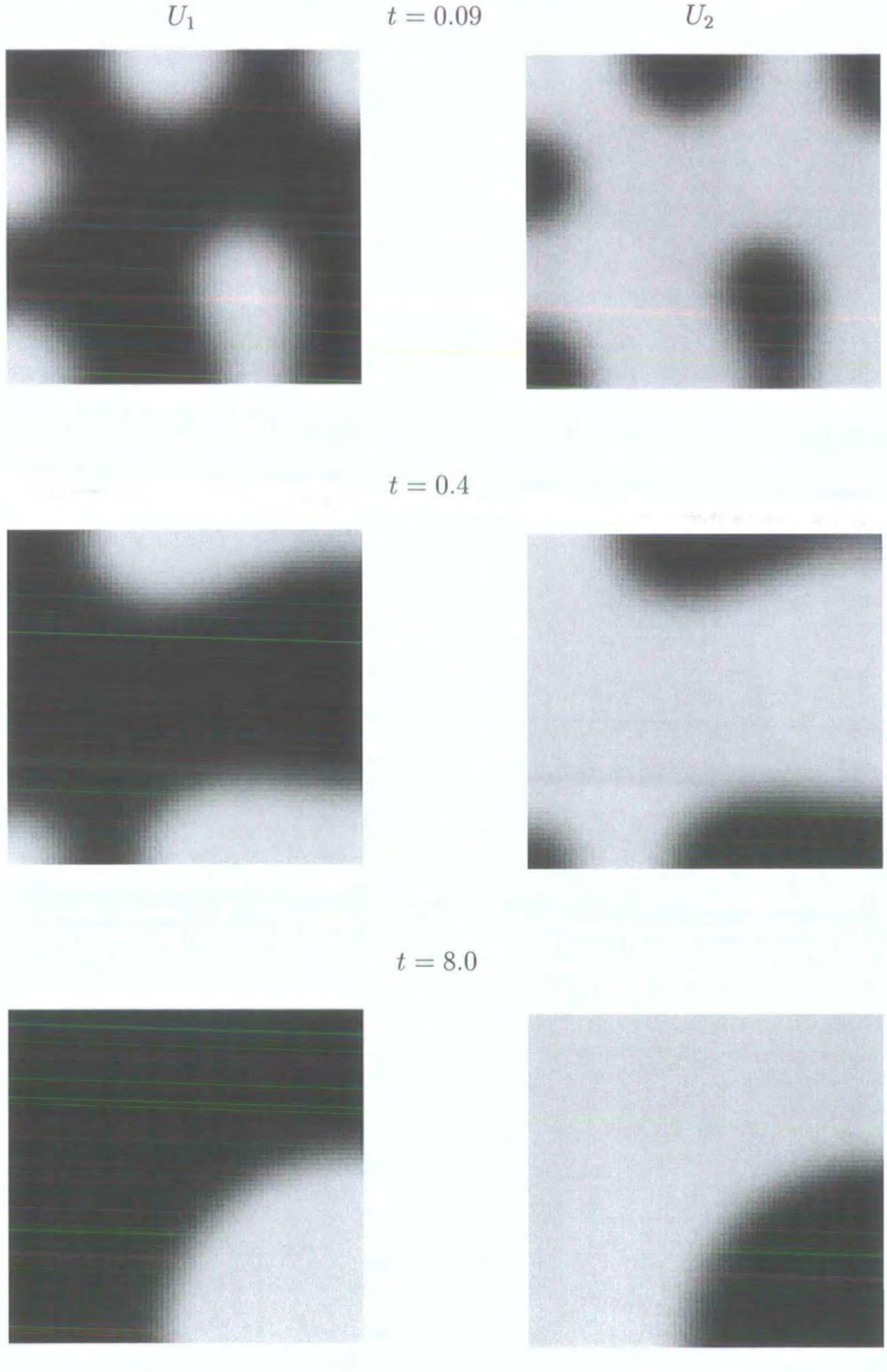

Figure 5.20: The early stage evolution from the initial conditions that are random perturbations of the uniform state $U_1 = 0.5$ and $U_2 = -0.5$ at $t = 0.09$, 0.4, 8.0.

(5.1.11a–b)).

To see a clear interaction between the solutions U_1 and U_2 in terms of their physical meaning, it would be worthwhile doing computational experiment in three space dimensions.

Chapter 6

Conclusions

It was shown using a Faedo-Galerkin approximation that there exists a unique solution for the coupled pair of Cahn-Hilliard equations modelling a phase separation on a thin film of binary liquid mixture coating substrate, which is wet by one component. This solution satisfies certain stability bounds. The regularity result at the end of Chapter 2 is essential for obtaining the error bound for the method proposed.

Some mathematical tools were developed for analysing a semi and fully discrete approximation. The existence, uniqueness, and stability bound for the semidiscrete finite element approximation were proven for $d = 1, 2, 3$. An error bound between the semidiscrete and continuous solution was given for $d = 1, 2, 3$.

Two types of fully discrete approximations, called Scheme 1 and Scheme 2, for solving the weak formulation were proposed. The existence and uniqueness of both schemes, for $d = 1, 2, 3$, were proven. Their stability estimates were shown for $d = 1, 2, 3$. The convergence of the solutions to the continuous problem in the weak formulation form for $d = 1, 2, 3$, was discussed for Scheme 1.

The error bound between the fully discrete and continuous solutions for Scheme 1 was proven by combining the error bound between the semi discrete approximation and continuous problem, and the fully and semi discrete approximation. The error bound is not optimal in the sense it linear in Δt. It might be possible to improve the error estimate as Barrett and Blowey did in [4–6], and we leave this for future work.

Two practical algorithms (implicit and explicit) for solving the finite element approximation at each time step were suggested. The convergence theory for the implicit scheme, which was used to solve the system arising from Scheme 1 was proven. The linear stability analysis for one space dimension was given. Simulations in one and two space dimensions were performed using the implicit scheme, and all computational results matched the linear stability region we have shown.

The analysis of (1.0.1a–e) would be greatly simplified if the prototype nonsmooth potential

$$F(u_1, u_2) = \begin{cases} \frac{1}{2}(1 - u_1^2) + \frac{1}{2}(1 - u_2^2) + \frac{\gamma}{2}|\nabla u_1|^2 + \frac{\gamma}{2}|\nabla u_2|^2 + D(u_1 + 1)(u_2 + 1), \\ +\infty \quad \text{for } |u_1| \geq 1 \quad \text{or } |u_2| \geq 1, \end{cases}$$

was used instead of (1.0.1e). This would lead to a pair of coupled variational inequalities and perhaps the work of Blowey and Elliott [9, 10] could be generalised. The advantage in this case would be that the only "nonlinear" term (for want of a better word) would be the variational inequality.

Modica in [30] consider a mathematical problem studying the asymptotic behaviour as $\gamma \to 0^+$ of solutions u_γ of the minimisation problem

$$\min \int_\Omega \gamma|\nabla u|^2 + \Psi(u)dx,$$

such that $\int_\Omega u(x)dx = m$. It is may be possible to mimic this study to analyse the asymptotic behaviour as $\gamma \to 0^+$ of the minimising free energy (2.2.14) which was mentioned in Section 5.3.1. This could be the basis of rigorous analysis for studying the behaviour of the solutions of our problem in Figure 5.14. We left this for future research.

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Appendix A

Basic and Auxiliary Results

A.1 Basic Results

Theorem A.1.1 (Lax-Milgram lemma) Let V be a Hilbert space, let $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ be a continuous V-elliptic bilinear form, and let $f : V \mapsto \mathbb{R}$ be a continuous linear form. Then the abstract variational problem: Find an element u such that

$$u \in V \quad \text{and} \quad \text{for all} \quad v \in V, \quad a(u,v) = f(v),$$

has one and only one solution.

Proof: See Ciarlet [13] page 8, for example.

Theorem A.1.2 (Compactness) Let V, H and V' be three Banach spaces with V and V' being reflexive and

$$V \subset H \equiv H' \subset V',$$

where the injection $V \hookrightarrow H$ is compact. Also let

$$W = \{v : v \in L^p(0,T;V), \frac{dv}{dt} \in L^q(0,T;V')\},$$

where $T < \infty$ and $1 < p, q < \infty$. Then the injection W in $L^p(0,T;H)$ is compact.

Proof: See Lions [27] page 58.

Theorem A.1.3 Let V, H and V' be three Hilbert spaces, having the property that

$$V \subset H \equiv H' \subset V'.$$

If $u \in L^2(0,T;V)$ and $u' \in L^2(0,T;V')$ then $u \in C([0,T];H)$ *a.e* and the following equality holds in the scalar distribution sense on $(0,T)$

$$\frac{d}{dt}|u|^2 = 2\langle u', u\rangle$$

Proof: See Temam [34] page 261.

Theorem A.1.4 (See Dautray and Lions [17] page 289) Let V be a reflexive Banach space, $\{\eta_n\}$ a bounded sequence in V. Then it is possible to extract from $\{\eta_n\}$ a subsequence which convergences weakly in V.

Theorem A.1.5 (See Dautray and Lions [17] page 291) Let V be a separable normed space and V' its dual. Then from every bounded sequence in V', it is possible to extract subsequence which is weak-star convergent in V'.

Theorem A.1.6 (Grönwall Inequality) Let C be a nonnegative constant and let u and v be continuous nonnegative functions on some interval $t \in [\alpha\,, \beta]$ satisfying the inequality

$$v(t) \leq C + \int_\alpha^t v(s)u(s)ds \quad \text{for } \ t \in [\alpha\,, \beta].$$

Then

$$v(t) \leq C \exp\Big(\int_\alpha^t u(s)ds\Big) \quad \text{for } \ t \in [\alpha\,, \beta].$$

Proof: See Brauer and Nohel [12] page 31 for example.